All-Pay Auctions with Different Forfeit Functions

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Abstract

A common method of selling items is via auction. In an auction, each bidder bids a certain amount of money, and the bidder bidding the most is the winner. The amount of money each bidder must pay is variable depending on the type of auction. These auctions can also be used as a model for other real world conflicts. In this work we focus on all-pay auctions and extend existing results in the literature for a generalized forfeit function. Using this model of an auction we can model a trade war between two countries. We will also outline future intentions to investigate the results of auctions with more forms of the forfeit function, different natures of bidders, and more prizes. These results would allow sellers to know the optimal auction in which to sell items and tell bidders the optimal bid they should make.

Keywords: All-pay Auction, Auction Forfeit Function, Economic Model, Tariff, US-China trade war

Contents

1 Introduction

For thousands of years, auctions have been used as a method for selling objects. There are four main types of auctions that are commonly used. The first type of auction is the English auction. In this type of auction, the seller continually raises the price of the item until only one person is willing to pay, and the item is sold at this price. A second type of auction is the Dutch auction. In this auction, the seller sets an extremely high price and continually lowers it until a bidder is willing to pay. A third type of auction is the first-price sealed-bid auction. In this type, bidders all bid simultaneously and the bidder with the highest bid wins and pays that bid. A fourth type of auction is the second-price sealed-bid auction, where bidders also bid simultaneously, and the bidder with the highest bid wins but pays the second-highest bid. These four types of auctions have already been analyzed extensively with the independent private values model. In this model, bidders are risk-neutral and only know the value of the object to himself. In this model, the first-price auction is equivalent to the Dutch auction, while the second-price auction is equivalent to the English auction. Another significant result relating to the independent private values model is the revenue equivalence theorem, which says that when information is independently and equally distributed, the revenue earned by the seller is constant regardless of the type of auction [5]. One last phenomenon associated with the independent values model is the winner's curse [6], which means that winner tend to overestimate the value of the object. This winner's curse is demonstrated in the field of oil. Petroleum engineers found that oil companies earned low profits due to this phenomenon [7].

Although the four auctions previously described are the most popular ones, there are some more variations. The next two types of auctions are the war of attrition and all-pay auction. In both of these, the bidders who don't win must pay the value of their bid. However, in the former, the winner pays the second-highest bid while in the latter, the winner pays the highest bid. There are also variations of the all-pay auction where the non-winning bidders pay different amounts based on all of the bids. One example is a constant entrance fee, or the bidders paying a fraction of their bid. These auctions are not as prominent, but are still useful. They can also be used as a model for many other systems. Conflicts among animals [3] can be represented by the war of attrition. All-pay auctions can be used to model the arms race [8] and wars [10].This type of auction can also model rent-seeking scenarios such as lobbying [4] or competition with sunk investments [9].

The two most notable papers in this field are by Milgrom and Weber [2] and Krishna and Morgan [1]. The paper by Milgrom analyzed the expected selling prices and bidding strategies of the second-price auction, English auction, and the first-price auction, as well as how they compare to each other when all bidders are risk-neutral. Furthermore, it reveals how the results of auctions are changed when new information is publicly revealed, a reserve price is set, or an entrance fee is set. Finally, the paper investigates when bidders are risk-averse rather than risk-neutral. A second prominent paper in this field is by Krishna and Morgan [1]. This paper expands on the results of Milgrom by calculating the bidder strategies and generated revenues of the war of attrition and all-pay auction. Then it ranks the revenues generated by these two auctions against those generated by the first and second-price auctions. Lastly, it compares the bidder's expected payoff for different types of auctions. There were also multiple other papers that analyzed all-pay auctions. Amann and Leininger analyzed the case with two bidders [15]. The behavior of bidders in an all-pay auction with incomplete information was studied in both [12] and [14]. Lastly, Che and Gale studied the relationship between all-pay and first-price auctions [13]. This paper will extend these works by generalizing these theorems and results to all-pay auctions with different forfeit functions.

In this paper, we extend the results of Krishna and Morgan [1] to all-pay auctions with different forfeit functions for all the losing bidders. Specifically, we will examine auctions where there is an entrance fee in addition to paying the bid, both when the fee is returned to the winner and when it is not. We will also examine when the forfeit function is a constant fraction of the original bid. For these auctions, we develop an expression for the symmetric bidding strategy in each type of auction. Then between these types of auctions, we compare the revenue made for the seller. The symmetric bidding equilibrium strategy that we found for this form of an auction can be used to model the ongoing US-China Trade War. We are working on quantifying the variables affecting the war. Then we can divide the war into timeslices, each with different values for variables to model the war. This model could provide guidance towards actions that should be taken to give the best possible result for the economies of the countries involved. We also develop approximations for an exponential forfeit as the bid grows larger.

2 Background on All-Pay Auctions

2.1 Generic Model for Auctions

Milgrom and Weber developed a model that can be used for any symmetric auction which we outline below. Suppose there are n bidders all bidding for a single object. Each bidder will have their own information about the object, so let $X = (X_1, X_2, \ldots, X_n)$ be a vector whose components are the informational variables known by each bidder. Then let $S = (S_1, S_2, \ldots S_m)$ be a vector containing additional variables that affect the opinion of bidders and may be known to the seller. We suppose there is a nonnegative finite function u such that $u(S, X_i, \{X_j\}_{j \neq i}) = V_i$, the value of the object to bidder i. We let the payoff of the winner be $V_i - b$ where b is the price paid.

Now we let $f(s, x)$ be the joint probability distribution of the random variables where f is symmetric in the last n variables. Furthermore, f follows the affiliation inequality, which says that $f(z \vee z')f(z \wedge z') \geq f(z)f(z')$ where $z \vee z'$ is the component-wise maximum and $z \wedge z'$ is the component-wise minimum. Now let $Y_1 = max\{X_j\}_{j \neq 1}$. This implies that it is more likely for the variables to be close to each other than farther apart.

Then let $f_{Y_1}(\cdot|x)$ be the conditional density of Y_1 if $X_1 = x$ and $F_{Y_1}(\cdot|x)$ be the corresponding cumulative distribution. The cumulative distribution of a function f at a point y is defined as the probability that the result is at most $f(y)$. This can alternatively be expressed as

$$F_{Y_1}(y|x) = \int_{-\infty}^{y} f_{Y_1}(s|x)ds.$$

Lastly, we define $v(x, y) = E[V_1|X_1 = x, Y_1 = y]$. This function v represents the expected value of the object to bidder 1.

2.2 Overview of Existing Papers

The model given above has been used to study both the all-pay auction in [1] and the first-price auction in [2]. In all-pay auctions, the payoffs are determined by [1]

$$W_i = \begin{cases} V_i - b_i & b_i > \max_{j \neq i} b_j \\ -b_i & b_i < \max_{j \neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i & b_i = \max_{j \neq i} b_j. \end{cases} \tag{1}$$

Now we begin by deriving a heuristic that is necessary for the symmetric equilibrium bidding strategy. Suppose bidders $j \neq 1$ follow the symmetric, increasing equilibrium strategy α. The expected payoff of bidder 1 where $X_1 = x$ and bids b is

$$\Pi(b, x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x, y) f_{Y_1}(y|x) dy - b. \tag{2}$$

We want to maximize this based on the bid, so we set the derivative with respect to b to 0 to get

$$v(x,\alpha^{-1}(b))f_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))}-1=0. \tag{3}$$

At symmetric equilibrium, bidder 1 must also follow the bidding strategy α, so $\alpha(x)=b$, which gives

$$\alpha(x)=b \implies \alpha'(x)=v(x,x)f_{Y_1}(x|x) \implies \alpha(x)=\int_{-\infty}^{x} v(t,t)f_{Y_1}(t|t)dt. \tag{4}$$

However, this only gives a necessary condition for the formula of the bidding strategy.

Theorem 1. *Let $\psi(x,y)=v(x,y)f_{Y_1}(y|x)$. If $\psi(x,y)$ is increasing in x, then the formula for symmetric equilibrium function is given by*

$$\alpha(x)=\int_{-\infty}^{x} v(t,t)f_{Y_1}(t|t)dt.$$

This theorem gives a function that represents the bidding strategy for bidders in an all-pay auction. A proof of this can be found in [1]

Theorem 2. *The function of symmetric equilibrium for a first-price auction is given by*

$$\alpha(x)=\int_{-\infty}^{x} v(s,s)\frac{f_{Y_1}(s|s)}{F_{Y_1}(s|s)}\exp\left(\int_{x}^{s}\frac{f_{Y_1}(t|t)}{F_{Y_1}(t|t)}dt\right)ds.$$

This theorem gives a function that represents the bidding strategy for bidders in a first-price auction. A proof of this can be found in [2]

Theorem 3. *If $\psi(x,y)$ is increasing in x, then the expected revenue from an all-pay auction is at least as great as that from a first-price auction.*

These theorems show the expected results of all-pay and first-price auctions. These results are important to sellers since it can help them determine what type of auction they should use and how much they should expect to receive.

3 Auctions with Constant Entrance Fees

In this section, we will investigate the effects of introducing a constant entrance fee to an all-pay auction. First, we will examine when the winner does not have his entrance fee returned, and then we will examine when he does.

3.1 Entrance Fee not Returned

For this case, the expected payoff is given by the following:

$$W_i = \begin{cases} V_i - b_i - c & b_i > \max_{j\neq i} b_j \\ -b_i - c & b_i < \max_{j\neq i} b_j \\ \frac{V_i}{\#\{k: b_k = b_i\}} - b_i - c & b_i = \max_{j\neq i} b_j. \end{cases} \quad (5)$$

We again derive a heuristic for the bidding strategy. Suppose bidders $j \neq 1$ follow symmetric increasing equilibrium strategy α. The expected payoff of bidder 1 where $X_1 = x$ and bids b is

$$\Pi(b, x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x, y) f_{Y_1}(y|x) dy - b - c. \quad (6)$$

We want to maximize this based on the bid, so we set the derivative with respect to b to 0 to get

$$v(x, \alpha^{-1}(b)) f_{Y_1}(\alpha^{-1}(b)|x) \frac{1}{\alpha'(\alpha^{-1}(b))} - 1 = 0. \quad (7)$$

However, this equation does not depend on c at all, which means that the addition of a constant entrance fee to an all-pay auction does not affect the strategy if the entrance fee is paid by everyone.

3.2 Entrance Fee Returned to the Winner

Now the expected payoff is given by this:

$$W_i = \begin{cases} V_i - b_i & b_i > \max_{j\neq i} b_j \\ -b_i - c & b_i < \max_{j\neq i} b_j \\ \frac{V_i}{\#\{k: b_k = b_i\}} - b_i & b_i = \max_{j\neq i} b_j. \end{cases} \quad (8)$$

Suppose bidders $j \neq 1$ follow symmetric increasing equilibrium strategy α. The expected payoff of bidder 1 where $X_1 = x$ and bids b is

$$\Pi(b, x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x, y) f_{Y_1}(y|x) dy - b - c(1 - F_{Y_1}(\alpha^{-1}(b)|x)). \quad (9)$$

We want to maximize this, so we set the derivative with respect to b to 0 to get

$$v(x, \alpha^{-1}(b)) f_{Y_1}(\alpha^{-1}(b)|x) \frac{1}{\alpha'(\alpha^{-1}(b))} - 1 + c \cdot f_{Y_1}(\alpha^{-1}(b)|x) \frac{1}{\alpha'(\alpha^{-1}(b))} = 0, which gives \quad (10)$$

$$
\begin{aligned}
&\alpha(x) = b \\
&\implies \alpha'(x) = (v(x,x)+c)f_{Y_1}(x|x) \\
&\implies \alpha(x) = \int_{-\infty}^{x} (v(t,t)+c)f_{Y_1}(t|t)dt \\
&\implies \alpha(x) = \int_{-\infty}^{x} v(t,t)f_{Y_1}(t|t)dt + c\int_{-\infty}^{x} f_{Y_1}(t|t)dt \\
&\implies \alpha(x) = \int_{-\infty}^{x} v(t,t)f_{Y_1}(t|t)dt + c\int_{-\infty}^{x} f_{Y_1}(t|t)dt.
\end{aligned} \tag{11}
$$

This shows that when the entrance fee is returned to the winner, the bidding strategy changes. This happens because there is no longer symmetry in the forfeits. Furthermore, when this symmetry is destroyed, the bidding amount increases when the entrance fee increases.

4 Auctions with Fractional Forfeits

We will now analyze the effects of having the forfeit be a fraction of the bid. The expected payoff is as follows:

$$
W_i = \begin{cases} V_i - b_i & b_i > \max_{j\neq i} b_j \\ -\beta b_i & b_i < \max_{j\neq i} b_j \\ \frac{V_i}{\#\{k: b_k = b_i\}} - b_i & b_i = \max_{j\neq i} b_j. \end{cases} \tag{12}
$$

Suppose bidders $j \neq 1$ follow symmetric increasing equilibrium strategy α. The expected payoff of bidder 1 where $X_1 = x$ and bids b is

$$
\Pi(b,x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x,y)f_{Y_1}(y|x)dy - bF_{Y_1}(\alpha^{-1}(b)|x) - (\beta b)(1 - F_{Y_1}(\alpha^{-1}(b)|x)). \tag{13}
$$

This is the form of the payoff since the integral represents the expected value of the object when bidder 1 wins and the rest of the terms represent the amount bidder 1 pays, depending on whether or not he won the auction.
We want to find the b that maximizes Π, so we set the derivative with respect to b to 0 to get

$$
\begin{aligned}
&v(x,\alpha^{-1}(b))f_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))} - F_{Y_1}(\alpha^{-1}(b)|x) - bf_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))} \\
&- \beta(1 - F_{Y_1}(\alpha^{-1}(b)|x)) + (\beta b)\cdot f_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))} = 0.
\end{aligned} \tag{14}
$$

We can multiply both sides of the equation by $\alpha'(\alpha^{-1}(b))$ and rearrange terms to get

$$\begin{aligned}&\alpha(x) = b\\ &\qquad \implies \beta\alpha'(x) + (1-\beta)\alpha'(x)F_{Y_1}(x|x) + (1-\beta)\alpha(x)f_{Y_1}(x|x) = v(x,x)f_{Y_1}(x|x).\end{aligned} \tag{15}$$

This becomes a first order differential equation in $\alpha(x)$. Solving this equation, we get

$$\begin{aligned}\alpha(x) &= \int_{-\infty}^{x} v(s,s)\frac{dL(s,x)}{1-\beta}\\ &\text{where } L(s,x) = \exp\left((1-\beta)\int_x^s \frac{f_{Y_1}(t|t)}{\beta + (1-\beta)F_{Y_1}(t|t)}dt\right).\end{aligned} \tag{16}$$

Notice that this can also be rewritten as

$$\alpha(x) = \int_{-\infty}^{x} v(s,s)\frac{f_{Y_1}(s|s)}{\beta + (1-\beta)F_{Y_1}(s|s)} \exp\left(-(1-\beta)\int_s^x \frac{f_{Y_1}(t|t)}{\beta + (1-\beta)F_{Y_1}(t|t)}dt\right) ds. \tag{17}$$

Notice that when $\beta = 0$, we get the strategy for first-price auctions, and when $\beta = 1$, we get the strategy for the original all-price auction.

Theorem 4. *When $\alpha(x)$ is as defined above, it is a symmetric equilibrium. Let $t(x) = v(x,x)$. We can use integration by parts to rewrite $\alpha(x)$ as*

$$\alpha(x) = \frac{v(x,x) - \int_{-\infty}^{x} L(s,x)dt(x)}{1-\beta}. \tag{18}$$

Proof. Notice that $L(\cdot|x)$ is decreasing and $v(x,x)$ is increasing, so by (21), $\alpha(x)$ is increasing. Now suppose $\alpha(x)$ is continuous. We can assume $\alpha(x)$ is differentiable without loss of generality by monotonically rescaling bidder estimates. We want to show that $\alpha(x)$ is the optimal bid when $X_1 = x$, which means

$$\frac{\partial}{\partial b}\Pi(\alpha(z);x) = \frac{f_{Y_1}(z|x)}{\alpha'(z)}\left(v(x,z) - (1-\beta)\alpha(z) - \alpha'(z)\frac{(1-\beta)F_{Y_1}(z|x)}{f_{Y_1}(z|x)}\right) - \beta. \tag{19}$$

Lemma 1: $\frac{F_{Y_1}(x|z)}{f_{Y_1}(x|z)}$ is decreasing in z.
Proof: By the affiliation inequality, for $\alpha \le x$ and $z' \le z$,

$$f_{Y_1}(\alpha|z)f_{Y_1}(x|z') \le f_{Y_1}(\alpha|z')f_{Y_1}(x|z) \implies \frac{f_{Y_1}(\alpha|z)}{f_{Y_1}(x|z)} \le \frac{f_{Y_1}(\alpha|z')}{f_{Y_1}(x|z')}. \tag{20}$$

Now we can integrate with respect to α from $-\infty$ to x to get

$$\frac{F_{Y_1}(x|z)}{f_{Y_1}(x|z)} \le \frac{F_{Y_1}(x|z')}{f_{Y_1}(x|z')}. \tag{21}$$

Now we can apply Lemma 1 and the fact that $v(x,z)$ is increasing to see that $\frac{\partial}{\partial b}\Pi(\alpha(z);x)$ has the same sign as $z-x$. This means $\Pi(\alpha(z);x)$ is maximized when $z=x$.
Now consider when α is discontinuous at a point x. This implies that for any positive ϵ, we have

$$\begin{aligned}
\infty &= \int_x^{x+\epsilon} \frac{(1-\beta)f_{Y_1}(s|s)}{\beta+(1-\beta)F_{Y_1}(s|s)} \\
&\leq \int_x^{x+\epsilon} \frac{f_{Y_1}(s|s)}{\beta F_{Y_1}(s|s)+(1-\beta)F_{Y_1}(s|s)} \\
&= \int_x^{x+\epsilon} \frac{f_{Y_1}(s|s)}{F_{Y_1}(s|s)} \\
&\leq \int_x^{x+\epsilon} \frac{f_{Y_1}(s|x+\epsilon)}{F_{Y_1}(s|x+\epsilon)} \\
&= \ln(F_{Y_1}(x+\epsilon|x+\epsilon)) - \ln(F_{Y_1}(x|x+\epsilon)).
\end{aligned} \tag{22}$$

However, for the second expression to be infinite, we need $F_{Y_1}(x|x+\epsilon)=0$. This case is just Theorem 14 in [2]. Therefore, the expression for $\alpha(x)$ given earlier is an equilibrium of this auction. □

Theorem 5. *The expected revenue generated for the seller of an all-pay auction with fractional cost is always less than when $\beta=1$ if $f(y|x)$ is increasing in x.*

Proof. Let $\alpha^\beta(x)$ be the equilibrium bid for a specific value of β. Notice that the expected payment of a bidder is

$$\begin{aligned}
&e_\beta(x) = (F_{Y_1}(x|x)+\beta(1-F_{Y_1}(x|x)))\alpha^\beta(x) \\
&= \int_{-\infty}^x v(s,s)f_{Y_1}(s|s)\frac{\beta+(1-\beta)F_{Y_1}(x|x)}{\beta+(1-\beta)F_{Y_1}(s|s)}\exp\left(-\int_x^s \frac{(1-\beta)f_{Y_1}(t|t)}{\beta+(1-\beta)F_{Y_1}(t|t)}dt\right)ds.
\end{aligned} \tag{23}$$

Since $f_{Y_1}(y|x)$ is increasing in x, $\frac{\beta}{f_{Y_1}(y|x)}$ is decreasing in x. Combined with Lemma 1, this means $\frac{f_{Y_1}(y|x)}{\beta+(1-\beta)F_{Y_1}(y|x)}$ is increasing in x. This means that

$$\begin{aligned}
-\int_s^x \frac{(1-\beta)f_{Y_1}(t|t)}{\beta+(1-\beta)F_{Y_1}(t|t)}dt &\leq -\int_s^x \frac{(1-\beta)f_{Y_1}(t|s)}{\beta+(1-\beta)F_{Y_1}(t|s)}dt \\
&= \ln(\beta+(1-\beta)F_{Y_1}(s|s)) - \ln(\beta+(1-\beta)F_{Y_1}(x|s)) \\
&\leq \ln(\beta+(1-\beta)F_{Y_1}(s|s)) - \ln(\beta+(1-\beta)F_{Y_1}(x|x))
\end{aligned} \tag{24}$$

where the last inequality comes from the fact that $F_{Y_1}(y|x)$ is non-increasing in x. This means that

$$e_\beta(x) \leq \int_{-\infty}^{x} v(s,s) f_{Y_1}(s|s) \frac{\beta + (1-\beta)F_{Y_1}(x|x)}{\beta + (1-\beta)F_{Y_1}(s|s)} \exp\left(\ln\left(\frac{\beta + (1-\beta)F_{Y_1}(s|s)}{\beta + (1-\beta)F_{Y_1}(x|x)}\right)\right)$$
$$\leq \int_{-\infty}^{x} v(s,s) f_{Y_1}(s|s) = e_1(x). \tag{25}$$

□

This shows that the expected amount paid by a bidder in an auction where $\beta \leq 1$ is at most the expected price paid by a bidder in the original all-pay auction. Since this is true about the bids of each bidder, it follows for the expected revenue earned by the seller as well.

In this section, we have proved the equilibrium bidding strategy for the all-pay auction with fractional forfeit. We also showed that each of these auctions does not generate as much revenue as the all-pay auction with complete bid forfeit. However, an ordering between two auctions with different values of β is yet to be determined.

5 Auctions with Exponential Forfeits

For this case, we will consider when the losers must pay an exponential forfeit. However, the differential equation we receive is difficult to solve, so we will consider when the bid is very large. The expected payoff is as follows:

$$W_i = \begin{cases} V_i - b_i & b_i > \max_{j\neq i} b_j \\ -e^{b_i} & b_i < \max_{j\neq i} b_j \\ \frac{V_i}{\#\{k : b_k = b_i\}} - b_i & b_i = \max_{j\neq i} b_j. \end{cases} \tag{26}$$

Suppose bidders $j \neq 1$ follow symmetric increasing equilibrium strategy α. The expected payoff of bidder 1 where $X_1 = x$ and bids b is

$$\Pi(b,x) = \int_{-\infty}^{\alpha^{-1}(b)} v(x,y) f_{Y_1}(y|x)dy - bF_{Y_1}(\alpha^{-1}(b)|x) - e^b(1 - F_{Y_1}(\alpha^{-1}(b)|x)). \tag{27}$$

This is the form of the payoff since the integral represents the expected value of the object when bidder 1 wins and the rest of the terms represent the amount bidder 1 pays, depending on whether or not he won the auction.

We want to find the b that maximizes Π, so we set the derivative with respect to b to 0 to get

$$\begin{aligned}v(x,\alpha^{-1}(b))f_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))}-F_{Y_1}(\alpha^{-1}(b)|x)-bf_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))}\\ -e^b(1-F_{Y_1}(\alpha^{-1}(b)|x))+e^b\cdot f_{Y_1}(\alpha^{-1}(b)|x)\frac{1}{\alpha'(\alpha^{-1}(b))}=0.\end{aligned} \tag{28}$$

We can multiply both sides of the equation by $\alpha'(\alpha^{-1}(b))$ and rearrange terms to get

$$\begin{aligned}&\alpha(x)=b\\ &\implies e^{\alpha(x)}\alpha'(x)+(1-e^{\alpha(x)})\alpha'(x)F_{Y_1}(x|x)+(\alpha(x)-e^{\alpha(x)})f_{Y_1}(x|x)=v(x,x)f_{Y_1}(x|x)\\ &\implies \alpha'(x)=\frac{(v(x,x)+e^{\alpha(x)}-\alpha(x))f_{Y_1}(x|x)}{e^{\alpha(x)}+(1-e^{\alpha(x)})F_{Y_1}(x|x)}.\end{aligned} \tag{29}$$

For large b,

$$\begin{aligned}\alpha'(x)&\approx\frac{f_{Y_1}(x|x)}{1-F_{Y_1}(x|x)}\\ \alpha(x)&\approx\int_{-\infty}^{x}\mathrm{d}t\left(\frac{f_{Y_1}(t|t)}{1-F_{Y_1}(t|t)}\right).\end{aligned} \tag{30}$$

Notice that this is independent of the function v, which represents the expected value of the object to bidder 1. As an example, consider when there are two bidders, so X denotes bidder 1's signal and Y denotes bidder 2's signal. Let $f(x,y)=\frac{4}{5}(1+xy)$ on $[0,1]x[0,1]$. This gives $f_{Y_1}(y|x)=\frac{2+2xy}{2+x}$ and $F_{Y_1}(y|x)=\frac{2y+xy^2}{2+x}$. Now we have

$$\alpha(x)=\int_0^x\frac{2+2t^2}{2-t-t^3}dt=\int_0^x\left(\frac{1}{1-t}-\frac{t}{2+t+t^2}\right)dt. \tag{31}$$

This function behaves very similarly to $-\ln(1-x)$ since the second term in the integral is negligible. Notice that this function increases slowly at first but then begins to increase more and more rapidly. This shows that with an exponential forfeit, the more likely winners bid significantly more than the less likely winners.

6 Summary and Future Work

In summary, we have investigated the effects of changing the forfeit function. We highlighted that the addition of a constant entrance fee does not affect the bidding strategy unless the fee is returned to the winner. When the forfeit is instead a fraction of the bid, we showed that the revenue generated by the seller is increasing with the fraction. Lastly, when the forfeit is exponential, the bidding strategy quickly approaches infinity.

In the future, we plan on analyzing the results of the all-pay auctions for more forms of the forfeit function, such as logarithmic, polynomial, or constant functions. We also will consider fractional forfeit where the forfeit is different for each place and the fraction is based on a distribution of the rational numbers. Using the fractional forfeit model, we will develop a model for the US-China trade war by dividing the war into timeslices, each an all-pay fractional forfeit auction. Furthermore, we will investigate the difference in results if the bidders are risk-averse rather than risk-neutral. Lastly, we will examine the effects of multiple prizes on the results of these auctions. The multiple prize all-pay auction has already been analyzed in [11], but we plan to extend it to other forfeits. These forms of auctions all exist in the real world, and therefore it is important to work towards fully understanding them.

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