New *Wilker-type Inequalities for Trigonometric Functions*

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Abstract

In this paper, we established two new *Wilker-Type inequalities for* trigonometric functions and proved the validity of such inequalities . *We have also given a concise proof of conventional Wilker's inequality and of* Hungens-type inequality.

Key words: *Wilker's inequalities,* Hungens-type inequality, *trigonometric Functions, power series expansion*

1 Introduction

In 1989, J.B.Wilker[2]proposed two open questions in the American Mathematical Monthly, among which the first one was:

Problem 1. If $0 < x < \frac{\pi}{2}$,then

$$(\frac{\sin x}{x})^2 + \frac{\tan x}{x} > 2 \tag{1.1}$$

the second one was:

Problem 2. If $0 < x < \frac{\pi}{2}$,find the largest constant c such that

$$(\frac{\sin x}{x})^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{1.2}$$

They have aroused remarkable interest of many mathematicians who conducted a huge number of researches upon this topic.

J.S. Sumner et al.[3] proved that the truthfulness of (1.1) and (1.2) resulted in the following theorem 1:

Theorem 1. If $0 < x < \frac{\pi}{2}$,then

$$\frac{16}{\pi^4} x^3 \tan x < (\frac{\sin x}{x})^2 + \frac{\tan x}{x} - 2 < \frac{8}{45} x^3 \tan x \qquad (1.3)$$

Furthermore, $\frac{16}{\pi^4}$ and $\frac{8}{45}$ are the best constants in(1.3).

Recently, Zhu[6] gave a new simple proof of inequalities(1.1), and Zhang and Zhu[4]gave a new *elementary proof of Wilker's inequalities*(1.3).Zhu[5] showed *some new Wilker-Type inequalities for circular and hyperbolic functions.* L.Zhu and Marija Nenezić[11]gave new approximation inequalities for circular functions.

Another inequality, the Huygens inequality [13], aroused our interest in the process of researching. Such an inequality asserts that

If $0 < x < \frac{\pi}{2}$, then

$$2(\frac{\sin x}{x}) + \frac{\tan x}{x} > 3 \qquad (1.4)$$

In recent years, lots of papers concerning Huegens inequality has arisen, including but not limited to Zhu's[15], in which he has shown some new inequalities of the Huygens-type for trigonometric and hyperbolic functions; Chen's[16], in which he has given some new inequalities of the Huygens-type for inverse trigonometric and inverse hyperbolic functions; and also Chen and Cheung's,[14] in which they have shown but have failed to demonstrate an exact proof of Wilker and Huygens type inequalities including the following

Theorem 2. If $0 < x < \frac{\pi}{2}$,then

$$\frac{3}{20} x^3 \tan x < 2(\frac{\sin x}{x}) + \frac{\tan x}{x} - 3 < \frac{16}{\pi^4} x^3 \tan x \qquad (1.5)$$

Furthermore, $\frac{16}{\pi^4}$ and $\frac{3}{20}$ are the best constants in (1.5).

Subsequently, we establish two new *Wilker-Type inequalities theorem* 3 *and theorem* 4----*the main results of this paper. We'll show a concise proof of Wilker's inequality* (1.3) along with a proof of (1.5) using similarly succinct methods.

2 Some Lemmas

Lemma 1 (see [12], P.20, P.23). For $n \geq 1$,we $(-1)^{n-1}B_{2n} > 0$, have where B_n $(n \in N)$ are a type of numbers called the Bernoulli Numbers, defined by the following formula :

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

Lemma 2(see [7-11]) let B_{2n} be the even-indexed Bernoulli numbers, $n \geq 1, n \in N$ then

$$\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+2)(2n+1)}{\pi^2} < \frac{|B_{2n+2}|}{|B_{2n}|} < \frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+2)(2n+1)}{\pi^2}$$

Lemma 3(see [12], P.23,[5]). We know that the power expansions of tangent function and cotangent function are the following

$$\tan x = \sum_{n=1}^{\infty} (2^{2n}-1)2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!} \qquad |x| < \frac{\pi}{2} \tag{2.1}$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} 2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!} \qquad 0 < |x| < \pi \tag{2.2}$$

So, we can get the power expansions for the following functions

$$\sec^2 x = (\tan x)' = \sum_{n=1}^{\infty} (2^{2n}-1)2^{2n}(2n-1)|B_{2n}|\frac{x^{2n-2}}{(2n)!} \qquad |x| < \frac{\pi}{2} \tag{2.3}$$

$$\csc^2 x = (-\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} 2^{2n}(2n-1)\left|B_{2n}\right|\frac{x^{2n-1}}{(2n)!} \quad 0<|x|<\pi \tag{2.4}$$

$$\cot^2 x = \csc^2 x - 1 = \frac{1}{x^2} + \sum_{n=1}^{\infty} 2^{2n}(2n-1)\left|B_{2n}\right|\frac{x^{2n-1}}{(2n)!} - 1 \quad 0<|x|<\pi \tag{2.5}$$

$$\csc 2x = \frac{1}{2x} + \frac{1}{2}\sum_{n=1}^{\infty} 2^{2n}(2^{2n}-2)\left|B_{2n}\right|\frac{x^{2n-1}}{(2n)!} \quad 0<|x|<\frac{\pi}{2} \tag{2.6}$$

The formula (2.6) holds true because of the existence of the equation as follows:

$$\csc 2x = \frac{1}{\sin 2x} = \frac{1}{2\sin x\cos x} = \frac{\sin^2 x + \cos^2 x}{2\sin x\cos x} = \frac{1}{2}(\tan x + \cot x)$$

□

$$\frac{\sin x}{\cos^3 x} = \frac{1}{2}(\sec^2 x)' = \frac{1}{2}\sum_{n=2}^{\infty}(2^{2n}-1)2^{2n}(2n-1)(2n-2)\left|B_{2n}\right|\frac{x^{2n-3}}{(2n)!} \quad |x|<\frac{\pi}{2} \tag{2.7}$$

$$\cot x\csc^2 x = -\frac{1}{2}(\cot^2 x)' = \frac{1}{x^3} - \frac{1}{2}\sum_{n=2}^{\infty}2^{2n}(2n-1)(2n-2)\left|B_{2n}\right|\frac{x^{2n-3}}{(2n)!} \quad 0<|x|<\pi \tag{2.8}$$

$$\csc x = \frac{1}{x} + \sum_{n=1}^{\infty}(2^{2n}-2)\left|B_{2n}\right|\frac{x^{2n-1}}{(2n)!} \quad 0<|x|<\frac{\pi}{2} \tag{2.9}$$

$$\cot x\csc x = (-\csc x)' = \frac{1}{x^2} - \sum_{n=1}^{\infty}(2n-1)(2^{2n}-2)\left|B_{2n}\right|\frac{x^{2n-2}}{(2n)!} \quad 0<|x|<\frac{\pi}{2} \tag{2.10}$$

$$\begin{aligned}\frac{1}{\sin^3 x} &= \frac{1}{x^3} + \frac{1}{2}\sum_{n=2}^{\infty}(2^{2n}-2)(2n-1)(2n-2)\left|B_{2n}\right|\frac{x^{2n-3}}{(2n)!} \\ &\quad + \frac{1}{2x} + \frac{1}{2}\sum_{n=1}^{\infty}(2^{2n}-2)\left|B_{2n}\right|\frac{x^{2n-1}}{(2n)!} \quad 0<|x|<\frac{\pi}{2}\end{aligned} \tag{2.11}$$

The formula (2.11) holds true because of the existence of the equation as follows:

$$\frac{1}{\sin^3 x} = \frac{1}{2}(-\csc x\cot x)' + \frac{1}{2}\csc x = \frac{1}{2}(\csc x)'' + \frac{1}{2}\csc x$$

□

3 main results of this paper

Theorem 3. If $0<x<\frac{\pi}{2}$,then

$$(\frac{\sin x}{x})^2+\frac{\tan x}{x}-2>\frac{8}{45}x^4\left(\frac{\tan x}{x}\right)^{\frac{6}{7}} \tag{3.1}$$

Holds true. Furthermore, $\frac{8}{45}$ is the best constant in (3.1).

Proof . Let $f(x)=\dfrac{\sin^2 x+x\tan x-2x^2}{x^6(\frac{\tan x}{x})^{\frac{6}{7}}}$ then

$$f'(x)=\frac{(\sin 2x+\tan x+x\sec^2 x-4x)x^6(\frac{\tan x}{x})^{\frac{6}{7}}-(\sin^2 x+x\tan x-2x^2)[x^6(\frac{\tan x}{x})^{\frac{6}{7}}]'}{x^{12}(\frac{\tan x}{x})^{\frac{12}{7}}}$$

where

$$[x^6(\frac{\tan x}{x})^{\frac{6}{7}}]'=6x^5(\frac{\tan x}{x})^{\frac{6}{7}}+\frac{6}{7}x^6(\frac{\tan x}{x})^{-\frac{1}{7}}\frac{x\sec^2 x-\tan x}{x^2}$$

thus

$$f'(x)=\frac{7x\tan x(\sin 2x+\tan x+x\sec^2 x-4x)-(\sin^2 x+x\tan x-2x^2)(36\tan x+6x\sec^2 x)}{7x^8(\frac{\tan x}{x})^{\frac{13}{7}}}$$

$$=\frac{g(x)}{7x^8(\frac{\tan x}{x})^{\frac{13}{7}}}$$

where

$$g(x)=7x\tan x(\sin 2x+\tan x+x\sec^2 x-4x)-(\sin^2 x+x\tan x-2x^2)(36\tan x+6x\sec^2 x)$$

$$=14x\sin^2 x-36\sin^2 x\tan x+12x^3\sec^2 x-35x\tan^2 x+x^2\sec^2 x\tan x+44x^2\tan x$$

$$=\sin^2 x(14x-36\tan x+12x^3\frac{1}{\sin^2 x\cos^2 x}-35x\sec^2 x+x^2\frac{1}{\sin x\cos^3 x}+44x^2\frac{1}{\sin x\cos x})$$

$$=\sin^2 x(14x-36\tan x+12x^3\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}-35x\sec^2 x+x^2\frac{\sin^2 x+\cos^2 x}{\sin x\cos^3 x}+88x^2\csc 2x)$$

$$=\sin^2 x(14x-36\tan x+12x^3\sec^2 x+12x^3\csc^2 x-35x\sec^2 x+x^2\frac{\sin x}{\cos^3 x}+90x^2\csc 2x)$$

$$=\sin^2 x\cdot s(x)$$

where

$$s(x)=14x-36\tan x+12x^3\sec^2 x+12x^3\csc^2 x-35x\sec^2 x+x^2\frac{\sin x}{\cos^3 x}+90x^2\csc 2x$$

By using (2.1)(2.3)(2.4)(2.6)(2.7),we can obtain

$$s(x)=14x-36\sum_{n=1}^{\infty}2^{2n}(2^{2n}-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}+12x^3\sum_{n=1}^{\infty}2^{2n}(2^{2n}-1)(2n-1)|B_{2n}|\frac{x^{2n-2}}{(2n)!}$$

$$+12x^3(\frac{1}{x^2})+12x^3\sum_{n=1}^{\infty}2^{2n}(2n-1)|B_{2n}|\frac{x^{2n-2}}{(2n)!}+\frac{x^2}{2}\sum_{n=2}^{\infty}2^{2n}(2^{2n}-1)(2n-1)(2n-2)|B_{2n}|\frac{x^{2n-3}}{(2n)!}$$

$$-35x\sum_{n=1}^{\infty}2^{2n}(2^{2n}-1)(2n-1)|B_{2n}|\frac{x^{2n-2}}{(2n)!}+90x^2(\frac{1}{2x})+45x^2\sum_{n=1}^{\infty}2^{2n}(2^{2n}-2)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=71x+\sum_{n=1}^{\infty}(24n2^{2n}+33\cdot 2^{2n}-90)2^{2n}|B_{2n}|\frac{x^{2n+1}}{(2n)!}+\sum_{n=2}^{\infty}2^{2n}(2^{2n}-1)(2n-1)(n-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$-\sum_{n=1}^{\infty}(70n+1)2^{2n}(2^{2n}-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=71x+\sum_{n=1}^{\infty}(24n2^{2n}+33\cdot 2^{2n}-90)2^{2n}|B_{2n}|\frac{x^{2n+1}}{(2n)!}+\sum_{n=2}^{\infty}2^{2n}(2^{2n}-1)(2n-1)(n-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$-71x-\sum_{n=2}^{\infty}(70n+1)2^{2n}(2^{2n}-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=\sum_{n=1}^{\infty}(24n2^{2n}+33\cdot 2^{2n}-90)2^{2n}|B_{2n}|\frac{x^{2n+1}}{(2n)!}+\sum_{n=2}^{\infty}(2n^2-73n)2^{2n}(2^{2n}-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=\sum_{n=2}^{\infty}[24(n-1)2^{2n-2}+33\cdot 2^{2n-2}-90)2^{2n-2}|B_{2n-2}|\frac{x^{2n-1}}{(2n-2)!}+\sum_{n=2}^{\infty}(2n^2-73n)2^{2n}(2^{2n}-1)|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=\sum_{n=2}^{\infty}[(24n2^{2n-2}+9\cdot 2^{2n-2}-90)(2n)(2n-1)\frac{|B_{2n-2}|}{|B_{2n}|}+4(2n^2-73n)(2^{2n}-1)]2^{2n-2}|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$=\sum_{n=2}^{\infty} a_n 2^{2n-2}|B_{2n}|\frac{x^{2n-1}}{(2n)!}=\sum_{n=3}^{\infty} a_n 2^{2n-2}|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$ (for $a_2=0, when \quad n=2$)

Where, $a_n=(24n2^{2n-2}+9\cdot 2^{2n-2}-90)(2n)(2n-1)\frac{|B_{2n-2}|}{|B_{2n}|}+4(2n^2-73n)(2^{2n}-1)$

Theorem 2 shall be correct if we can successfully prove the following inequality:

$a_n>0, when \quad n\ge 3$

According to lemma 2, we have

$$\frac{|B_{2n-2}|}{|B_{2n}|}>\frac{2^{2n}-1}{2^{2n-2}-1}\cdot\frac{\pi^2}{(2n)(2n-1)}$$

So

$$a_n>(24n2^{2n-2}+9\cdot 2^{2n-2}-90)(2n)(2n-1)\frac{2^{2n}-1}{2^{2n-2}-1}\cdot\frac{\pi^2}{(2n)(2n-1)}+4(2n^2-73n)(2^{2n}-1)$$

$$=(24n2^{2n-2}+9\cdot 2^{2n-2}-90)\frac{2^{2n}-1}{2^{2n-2}-1}\cdot\pi^2+4(2n^2-73n)(2^{2n}-1)$$

$$=\frac{2^{2n}-1}{2^{2n-2}-1}[24n\pi^2 2^{2n-2}+9\pi^2\cdot 2^{2n-2}-90\pi^2+4(2n^2-73n)(2^{2n-2}-1)]$$

$=\frac{2^{2n}-1}{2^{2n-2}-1}b_n$, where,

$$b_n=[24n\pi^2 2^{2n-2}+9\pi^2\cdot 2^{2n-2}-90\pi^2+4(2n^2-73n)(2^{2n-2}-1)]$$

$$=24n\pi^2 2^{2n-2}+9\pi^2\cdot 2^{2n-2}+8n^2\cdot 2^{2n-2}+282n-282n\cdot 2^{2n-2}-8n^2-90\pi^2$$

$$b_3=11890-11610>0$$

$$b_n=(24n\pi^2+8\pi^2+8n^2-282n)2^{2n-2}+(282n-90\pi^2)+(\pi^2\cdot 2^{2n-2}-8n^2)$$

$=c_n 2^{2n-2}+(282n-90\pi^2)+(\pi^2\cdot 2^{2n-2}-8n^2)$,

where

$$c_n=24n\pi^2+8\pi^2+8n^2-282n$$

When $n>3$, $(282n-90\pi^2)>0, (\pi^2\cdot 2^{2n-2}-8n^2)>0$

And when $n \ge 9$ $c_n = 24n\pi^2 + 8\pi^2 + 8n^2 - 282n > 216n + 8n^2 - 282n = (8n-66)n > 0$

We can easily obtain

$c_4 = 104\pi^2 - 1000 \approx 1025 - 1000 > 0$

$c_5 = 128\pi^2 - 1210 \approx 1262 - 1210 > 0$

$c_6 = 152\pi^2 - 1404 \approx 1498 - 1404 > 0$

$c_7 = 176\pi^2 - 1582 \approx 1735 - 1582 > 0$

$c_8 = 300\pi^2 - 512 > 2700 - 512 > 0$

So $b_n > 0$ when $n \ge 3$,of cause, $a_n > 0$ when $n \ge 3$.

As we can see ,all coefficients of the polynomial $s(x)$ are positive integers. When $x > 0, s(x) > 0$, $g(x) = \sin^2 x \cdot s(x) > 0$.

when $x > 0,\ f'(x) = \dfrac{g(x)}{7x^8 (\frac{\tan x}{x})^{\frac{13}{7}}} > 0$

we can conclude that *f*(*x*) is strictly increasing on $(0, \frac{\pi}{2})$

$$so \quad f(x) > \lim_{x \to 0^+} f(x)$$

Furthermore $\lim_{x \to 0^+} f(x) = \frac{8}{45}$,and the proof of Theorem3 is complete.

Theorem 4. If $0 < x < \frac{\pi}{2}$,then

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 > \frac{2}{45} x^4 \left(\frac{\sin x}{x}\right)^{\frac{6}{7}} \qquad (3.2)$$

Holds. Furthermore, $\frac{2}{45}$ is the best constant in (3.2).

Proof.

Let

$$f(x)=\frac{x^2+\frac{x}{2}\sin 2x-2\sin^2 x}{x^6(\frac{\sin x}{x})^{\frac{20}{7}}}$$

Easy to find that

$$f'(x)=\frac{(2x+\frac{1}{2}\sin 2x+x\cos 2x-2\sin 2x)x^6(\frac{\sin x}{x})^{\frac{20}{7}}-(x^2+\frac{x}{2}\sin 2x-2\sin^2 x)[x^6(\frac{\sin x}{x})^{\frac{20}{7}}]'}{x^{12}(\frac{\sin x}{x})^{\frac{40}{7}}}$$

Where

$$[x^6(\frac{\sin x}{x})^{\frac{20}{7}}]'=6x^5(\frac{\sin x}{x})^{\frac{20}{7}}+x^6\frac{20}{7}(\frac{\sin x}{x})^{\frac{13}{7}}\frac{x\cos x-\sin x}{x^2}$$

Thus it can be reasoned that $f'(x)=\dfrac{g(x)}{7x^{10}(\frac{\sin x}{x})^{\frac{41}{7}}}$

where

$$g(x)=7x\sin^3 x(2x+x\cos 2x-\frac{3}{2}\sin 2x)-(x^2+\frac{x}{2}\sin 2x-2\sin^2 x)(22\sin^3 x+20x\sin^2 x\cos x)$$

$$=-21x^2\sin^3 x+6x^2\sin^5 x-3x\sin^4 x\cos x-20x^3\sin^2 x\cos x+44\sin^5 x$$

$$=\sin^5 x(6x^2+44-21x^2\csc^2 x-3x\cot x-20x^3\cot x\csc^2 x)$$

$$=\sin^5 x\cdot h(x)$$

where $h(x)=6x^2+44-21x^2\csc^2 x-3x\cot x-20x^3\cot x\csc^2 x$

By using(2.2)(2.4)(2.8),we can get

$$h(x)=6x^2+44-21x^2(\frac{1}{x^2}+\sum_{n=1}^{\infty}2^{2n}(2n-1)|B_{2n}|\frac{x^{2n-2}}{(2n)!})-3x(\frac{1}{x}-\sum_{n=1}^{\infty}2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!})$$

$$+20x^3(-\frac{1}{x^3}+\frac{1}{2}\sum_{n=2}^{\infty}2^{2n}(2n-1)(2n-2)|B_{2n}|\frac{x^{2n-3}}{(2n)!})$$

$$=6x^2+\sum_{n=1}^{\infty}(24-42n)2^{2n}|B_{2n}|\frac{x^{2n}}{(2n)!})+10\sum_{n=2}^{\infty}2^{2n}(2n-1)(2n-2)|B_{2n}|\frac{x^{2n}}{(2n)!})$$

$$=6x^2-6x^2+\sum_{n=2}^{\infty}(24-42n)2^{2n}|B_{2n}|\frac{x^{2n}}{(2n)!})+10\sum_{n=2}^{\infty}2^{2n}(2n-1)(2n-2)|B_{2n}|\frac{x^{2n}}{(2n)!})$$

$$=\sum_{n=2}^{\infty}a_n2^{2n}|B_{2n}|\frac{x^{2n}}{(2n)!}=\sum_{n=3}^{\infty}a_n2^{2n}|B_{2n}|\frac{x^{2n}}{(2n)!}$$

Where

$$a_n=10(2n-1)(2n-2)-42n+24=40n^2-102n+44$$

Let

$$a(x)=40x^2-102x+44$$
$$a'(x)=80x-102$$

Therefore, $a'(x)=80x-102>0$ holds true for any x such that $x>2$.

This implies that $a(x)$ is strictly increasing on $(2,+\infty)$.

Thus, when $x\in(2,+\infty)$, $a(x)>a(2)=0$

Which demonstrates that for all n>2,$a_n>0$.

Given the fact that all the coefficients of $h(x)$ are positive integers, $h(x)>0$ is true for every $x\in(0,\frac{\pi}{2})$, which would in turn prove that $g(x)>0$.

And as $7x^{10}(\frac{\sin x}{x})^{\frac{41}{7}}$ is surely greater than zero, this would indicate that $f'(x)$, while $x\in(0,\frac{\pi}{2})$, is also positive.

we can conclude that $f(x)$ is strictly increasing on $(0,\frac{\pi}{2})$

So, $f(x) > \lim_{x\to 0^+} f(x)$

Furthermore $\lim_{x\to 0^+} f(x) = \frac{2}{45}$,The proof of Theorem 4 is complete.

4 A Concise Proof of Theorem 1 and Theorem 2

4.1. A Concise Proof of Theorem1

Let $f(x) = \frac{\sin^2 x}{x^5 \tan x} + \frac{1}{x^4} - \frac{2}{x^3 \tan x}$

then $f'(x) = \frac{g(x)}{x^6 \sin^2 x}$

where $g(x) = 2x^3 + 6x^2 \sin x \cos x - 5\sin^3 x \cos x - 3x\sin^2 x - 2x\sin^4 x$

Direct calculation yields $g'(x) = 2\sin^3 x \cos x h(x)$

where $h(x) = 6x^2 \cot x \csc^2 x + 3x\csc^2 x - 9\cot x - 4x$

By using the power series expansion of (2.2)(2.4)(2.8),

$$h(x) = 6x^2[\frac{1}{x^3} - \frac{1}{2}\sum_{n=2}^{\infty}(2n-1)(2n-2)2^{2n}|B_{2n}|\frac{x^{2n-3}}{(2n)!}]$$

$$+3x[\frac{1}{x^2} + \sum_{n=1}^{\infty}(2n-1)2^{2n}|B_{2n}|\frac{x^{2n-2}}{(2n)!}]$$

$$-9[\frac{1}{x} - \sum_{n=1}^{\infty}2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!}] - 4x$$

$$= \sum_{n=2}^{\infty}[3-(2n-1)(2n-3)]3\cdot 2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!}$$

$$= \sum_{n=3}^{\infty}[3-(2n-1)(2n-3)]3\cdot 2^{2n}|B_{2n}|\frac{x^{2n-1}}{(2n)!} < 0$$

since $g'(x) = 2\sin^3 x \cos x h(x)$, when $x \in (0, \frac{\pi}{2}), \sin^3 x \cos x > 0$

So, $g'(x) = 2\sin^3 x \cos x h(x) < 0$

Then $g(x)$ is decreasing on $(0,\frac{\pi}{2})$. Now $g(0)=0$, so, $g(x)<0$

Therefore, $f'(x) = \frac{g(x)}{x^6 \sin^2 x} < 0$

So, $f(x)$ is strictly decreasing as x increases on $(0,\frac{\pi}{2})$.

At the same time,we find $\lim_{x\to 0^+} f(x) = \frac{8}{45}$ and $\lim_{x\to \frac{\pi}{2}^-} f(x) = \frac{16}{\pi^4}$. then the proof of theorem1 is complete.

4.2.A Concise Proof of Theorem2

Let $f(x) = \frac{2\sin x + \tan x - 3x}{x^4 \tan x}$

Thus the derivative of such a function could be expressed as follows:

$$f'(x) = \frac{(2\cos x + \sec^2 x - 3)x^4 \tan x - (2\sin x + \tan x - 3x)(4x^3 \tan x + x^4 \sec^2 x)}{x^8 \tan^2 x}$$

$$= \frac{1}{x^5 \tan^2 x} \cdot g(x)$$

where $g(x) = 2x\sin x - 8\sin x \tan x - 4\tan^2 x + 9x\tan x - 2x\sin x \sec^2 x + 3x^2 \sec^2 x$

$$= \frac{\sin^3 x}{\cos^2 x} h(x)$$

And where $h(x) = -2x - 8\cot x - 4\csc x + 9x\cot x \csc x + 3x^2 \frac{1}{\sin^3 x}$

By using the power series expansion of (2.2)(2.9)(2.10)(2.11), we can get

$$h(x) = -2x + 8\left(-\frac{1}{x} + \sum_{n=1}^{\infty} 2^{2n} |B_{2n}| \frac{x^{2n-1}}{(2n)!}\right) - 4\left(\frac{1}{x} + \sum_{n=1}^{\infty} (2^{2n}-2) |B_{2n}| \frac{x^{2n-1}}{(2n)!}\right)$$

$$+ 9x\left(\frac{1}{x^2} - \sum_{n=1}^{\infty} (2^{2n}-2)(2n-1) |B_{2n}| \frac{x^{2n-2}}{(2n)!}\right)$$

$$+ 3x^2\left(\frac{1}{x^3} + \frac{1}{2}\sum_{n=2}^{\infty} (2^{2n}-2)(2n-1)(2n-2) |B_{2n}| \frac{x^{2n-3}}{(2n)!}\right)$$

$$+ 3x^2\left(\frac{1}{2x} + \frac{1}{2}\sum_{n=1}^{\infty} (2^{2n}-2) |B_{2n}| \frac{x^{2n-1}}{(2n)!}\right)$$

$$= -\frac{1}{2}x + \sum_{n=1}^{\infty} [8 \cdot 2^{2n} - 4(2^{2n}-2) - 9(2n-1)(2^{2n}-2)] |B_{2n}| \frac{x^{2n-1}}{(2n)!}$$

$$+ \frac{3}{2}\sum_{n=2}^{\infty} (2^{2n}-2)(2n-1)(2n-2) |B_{2n}| \frac{x^{2n-1}}{(2n)!} + \frac{3}{2}\sum_{n=1}^{\infty} (2^{2n}-2) |B_{2n}| \frac{x^{2n+1}}{(2n)!}$$

$$= -\frac{1}{2}x + \frac{1}{2}x + \sum_{n=2}^{\infty} [8 \cdot 2^{2n} - 4(2^{2n}-2) - 9(2n-1)(2^{2n}-2)] |B_{2n}| \frac{x^{2n-1}}{(2n)!}$$

$$+ \frac{3}{2}\sum_{n=2}^{\infty} (2^{2n}-2)(2n-1)(2n-2) |B_{2n}| \frac{x^{2n-1}}{(2n)!} + \frac{3}{2}\sum_{n=2}^{\infty} (2^{2n-2}-2)(2n)(2n-1) |B_{2n-2}| \frac{x^{2n-1}}{(2n)!}$$

$$= \sum_{n=2}^{\infty} [8 \cdot 2^{2n} - 4(2^{2n}-2) - 9(2n-1)(2^{2n}-2) + \frac{3}{2}(2^{2n}-2)(2n-1)(2n-2) + \frac{3}{2}(2^{2n-2}-2)(2n)(2n-1)\frac{|B_{2n-2}|}{|B_{2n}|}] |B_{2n}| \frac{x^{2n-1}}{(2n)!}$$

$$= \sum_{n=2}^{\infty} d_n |B_{2n}| \frac{x^{2n-1}}{(2n)!}$$

Where,

$$d_n = 8 \cdot 2^{2n} - 4(2^{2n}-2) - 9(2n-1)(2^{2n}-2) + \frac{3}{2}(2^{2n}-2)(2n-1)(2n-2) + \frac{3}{2}(2^{2n-2}-2)(2n)(2n-1)\frac{|B_{2n-2}|}{|B_{2n}|}$$

$$= 4 \cdot 2^{2n} + 8 + 3(n-4)(2^{2n}-2)(2n-1) + 3n(2^{2n-2}-2)(2n-1)\frac{|B_{2n-2}|}{|B_{2n}|}$$

By careful calculation, $d_2 = 0, d_3 = 216 > 0$ can be gained.

When $n \geq 4, d_n > 0$ is true.

Then it can be reasonably obtained that

$$h(x) = \sum_{n=2}^{\infty} d_n |B_{2n}| \frac{x^{2n-1}}{(2n)!} = \sum_{n=3}^{\infty} d_n |B_{2n}| \frac{x^{2n-1}}{(2n)!} > 0$$

And thus $g(x) = \frac{\sin^3 x}{\cos^2 x} h(x) > 0$

As we know $f'(x) = \frac{1}{x^5 \tan^2 x} \cdot g(x) > 0$

So, $f(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$,

At the same time, we find $\lim_{x \to 0^+} f(x) = \frac{3}{20}$ and $\lim_{x \to \frac{\pi}{2}^-} f(x) = \frac{16}{\pi^4}$.

Now, the proof of theorem2 is complete.

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