Ricci-flat 5-regular graphs

Heidi Lei, Choate Rosemary Hall
Advisor: Shuliang Bai

September 9, 2019

Abstract

The notion of Ricci curvature of Riemannian manifolds in differential geometry has been extended to other metric spaces such as graphs. The Ollivier-Ricci curvature between two vertices of a graph can be seen as a measure of how closely connected the neighbors of the vertices are compared to the distance between them. A Ricci-flat graph is then a graph in which each edge has curvature 0. There has been previous work in classifying Ricci-flat graphs under different definitions of Ricci curvature, notably graphs with large girth and small degrees under the definition of Lin-Lu-Yau, which is a modification of Ollivier's definition of Ricci curvature. In this paper, we continue the effort of classifying Ricci-flat graphs and study specifically Ricci-flat 5-regular graphs under the definition of Lin-Lu-Yau.

Contents

1 Introduction

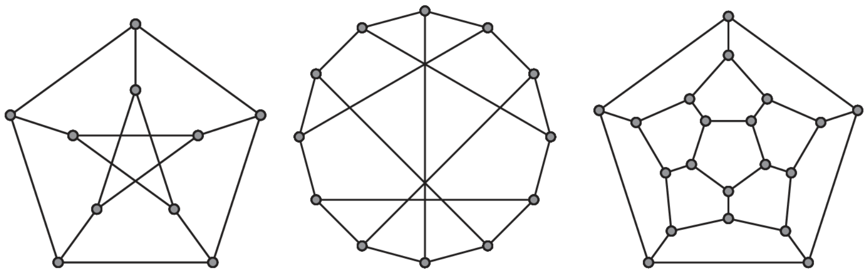
Ricci curvature is an important concept in differential geometry with wide applications in theoretical physics, such as general relativity and superstring theory. Essentially, Ricci curvature measures the amount of deviation in the volume of a section of a geodesic ball in a Riemannian manifold compared to its counterpart in Euclidean space. Naturally, a Ricci-flat manifold is a Riemannian manifold in which the Ricci curvature vanishes everywhere. They hold significance in physics as they represent vacuum solutions to the analogues of Einstein's equations generalized to Riemannian manifolds. One special class of Ricci-flat manifolds is Calabi-Yau manifolds, whose existence was conjectured by E. Calabi and proved by S.-T. Yau. There has been ongoing research in determining and analyzing the structures of Ricci-flat manifolds. One branch of such studies attempt to generalize the notion of Ricci curvature to other metric spaces, including discrete settings, such that analogues of important results in Riemannian manifolds such as Bonnet-Myers theorem hold.

Bakry-Emery-Ricci curvature generalizes Ricci curvature by defining a diffusion process on the manifold, and it has been studied on graphs in [3] and [9]. Y. Ollivier defines a sense of Ricci curvature using transportation distance and Markov chains on metric spaces including graphs in [10] and [11]. Ollivier-Ricci curvature on graph captures the idea that curvature describes the average distance between points inside small balls compared to the distance between their centers by distributing masses on a vertex and its neighbors, transferring the mass to another vertex and its neighbors, and calculating the transportation distance between the two vertices using an optimal transport plan. Ollivier-Ricci curvature is parametrized by its idleness, the amount of mass placed on the vertex themselves. The rest of the mass is distributed evenly among its neighbors. The Ollivier-Ricci curvature that is most studied is when the idleness is 0. Y. Lin, L. Lu, and S.-T. Yau modified Ollivier's definition of Ricci curvature to be the negative derivative when the idleness approaches 1 in Ollivier's definition, thus eliminating the idleness parameter [8]. With the modified Lin-Lu-Yau-Ricci curvature, they were able to study the Ricci curvature of Cartesian product graphs, random graphs, and other special classes of graphs.

[2] studied the Ollivier-Ricci curvature of graphs as a function of the chosen idleness parameter and showed that this idleness function is concave and piece-wise linear with at most 3 linear parts on its domain [0,1], with at most 2 linear parts in the case of a regular graph. Therefore, the Lin-Lu-Yau-Ricci curvature is equivalent to the negative of the slope of the last linear piece of the idleness function.

The problem of classifying Ricci-flat graphs under Lin-Lu-Yau's definition has been tackled through different angles and additional constraints. [7] classified Ricci-flat graphs with girth at least 5. [4] classified Ricci-flat cubic graphs of girth 5. [6] constructed an infinite family of distinct Ricci-flat graphs of girth four with edge-disjoint 4-cycles and completely characterize all Ricci-flat graphs of girth four with vertex-disjoint 4-cycles. [1] classified Ricci-flat graphs with maximum degree at most 4. The previous results on the classification of Ricci-flat regular graphs of small degree under Lin-Lu-Yau's definition is summarized below:

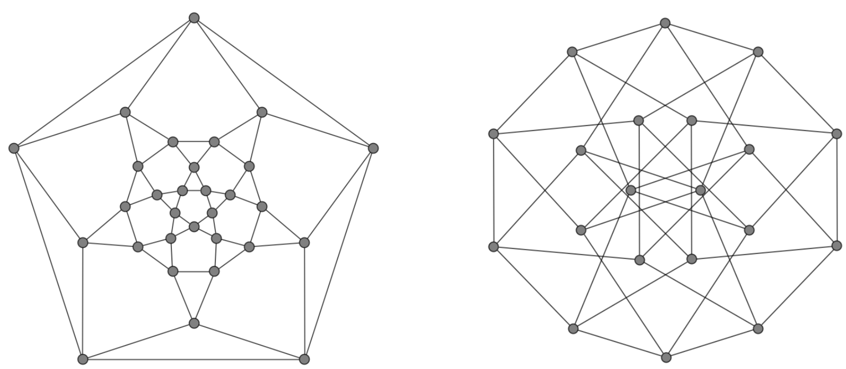
1. The Ricci flat 2-regular graphs are isomorphic to the infinite path and the cycle graph C_n with $n \geq 6$.
2. The Ricci flat 3-regular graphs are isomorphic to the Petersen graph, the Triplex graph and the dodecahedral graph.



(A) Petersen graph (B) Triplex (C) Dodecahedral graph

Figure 1.1: Ricci-flat 3-regular graphs.

3. The Ricci flat 4-regular graphs are isomorphic to one of two finite graphs: the icosidodecahedral graph and G_{20}; or are isomorphic to infinitely many lattice-type graphs in the terms of [1] in which each graph is locally a 4-regular grid.



(a) Icosidodecahedral graph (b) G_{20}

Figure 1.2: Ricci-flat 4-regular graphs.

[7] showed that Cartesian products of Ricci-flat regular graphs are Ricci-flat with the following theorem.

Theorem 1.1. *[7] Suppose that G is d_G-regular and H is d_H-regular. Then the Ricci curvature of $G \square H$ is given by*

$$\kappa^{G\square H}((u_1, v), (u_2, v)) = \frac{d_G}{d_G + d_H}\kappa^G(u,u_2),$$

$$\kappa^{G\square H}((u, v_1), (u, v_2)) = \frac{d_H}{d_G + d_H}\kappa^H(v,v_2)$$

where $u \in V(G), v \in V(H), u_1u_2 \in E(G)$, and $v_1v_2 \in E(H)$.

Corollary 1.1.1. *If both G and H are Ricci-flat regular graphs, so is the Cartesian product graph $G \square H$.*

Therefore, one class of Ricci-flat 5-regular graphs is the Cartesian product of a Ricci-flat 3-regular graph and a Ricci-flat 2-regular graph. As shown by [7] and [1], the Ricci-flat 3-regular graph has girth 5 and is either the Petersen graph, the Triplex graph, or the dodecahedral graph. The Ricci-flat 2-regular graph is either the cycle of length at least six or the infinite path.

1.1 Roadmap and main results

In this paper, we study Ricci-flat 5-regular graphs that are not of the Cartesian product type. Specifically, we have obtained the following main results.

In Section 2, we formalize the definition of Ricci curvature on graphs outlined in the introduction following the notations of Lin-Lu-Yau in [8].

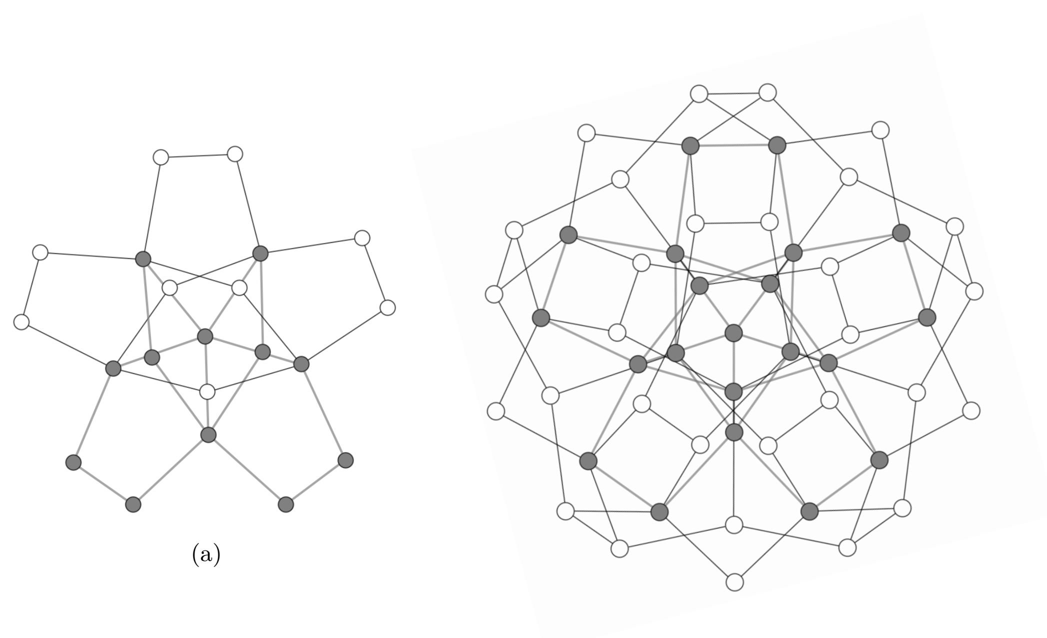
In Section 3, we analyze the local structure of a 5-regular graph by proving a more general result concerning regular graphs. Deferring the definition of local characteristics to Section 3, Lemma 3.1 essentially determines the Ricci curvature of an edge in a regular graph given its local environment. As an easy corollary, the local structure of any Ricci-flat regular graph can be determined by letting $\kappa(x, y) = 0$. There are five possible sets of local characteristics for a Ricci-flat 5-regular graph, and refer to them by type-A to type-E. See Fig. 3.1 for a schematic representation of the local structure of the edges.

Lemma 3.1. Let xy be an edge in a d-regular graph G with local characteristics (N_0, N_1, N_2). Then the Ricci-curvature of the edge xy is given by

$$\kappa(x, y) = -2 + \frac{4 + 3N_0 + 2N_1 + N_2}{d}.$$

Corollary 3.1.1. Let xy be an edge in a Ricci-flat 5-regular graph G. Then the local characteristics (N_0, N_1, N_2) of edge xy must be one of the following five types listed in Table 3.1.

In Section 4, we restrict our attention to symmetric graphs and found that Ricci-flat 5-regular symmetric graph must be isomorphic to a 5-regular symmetric graph of order 72, which we denote RF^5_{72}. Fig. 1.3 shows the subgraph induced by 2-neighborhood and 3-neighborhood of a vertex in RF^5_{72}, i.e. the subgraph induced by all vertices within a distance of 2 and 3, respectively, from the central vertex. The type-E local structure of an edge is highlighted in (a) and the the 2-neighborhood graph of RF^5_{72} shown in (a) is highlighted in (b). An adjacency list for RF^5_{72} can be found in the appendix.



(a) (b)

Figure 1.3: Ricci-flat 5-regular symmetric graph of order 72

Theorem 4.1. If G is a Ricci-flat 5-regular symmetric graph, then G is isomorphic to RF^5_{72}.

In Section 5, we turn our attention to the structure of more general Ricci-flat 5-regular graphs that are not and necessarily symmetric. When the symmetry condition is not imposed, the possible cases for the construction of the graph grow enormously. The main difficulty of such a classification lies in the lack of leverageable symmetries. We attack the problem by proving the nonexistence of certain substructures of a Ricci-flat 5-regular graph.

Lemma 5.6. If G is a Ricci-flat 5-regular graph, then it does not contain three adjacent triangles, i.e., the subgraph shown in Fig. 1.4.

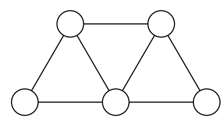


Figure 1.4

The following lemma asserts that there does not exist a Ricci-flat 5-regular graph that consists of only type-A, type-B, and type-C edges.

Lemma 5.9. If G is a Ricci-flat 5-regular graph, then it contains edges that are not in any triangle.

In Section 6, we give some conjectures on the classification of general Ricci-flat 5-regular graphs.

2 Notations and definitions

Let $G = (V, E)$ represent an undirected connected graph with vertex set V and edge set E without multiple edges or self loops. A vertex y is a neighbor of x if $xy \in E$. For a vertex $x \in V$, we denote the neighbors of x as $\Gamma(x)$ and the degree of x, i.e. the number of its neighbors, as d_x. If two vertices x, y are neighbors, we use $x \sim y$ to represent this relation. Let C_n represent a cycle of length n.

Definition 2.1. A probability distribution over the vertex set $V(G)$ is a mapping $\mu : V \to [0, 1]$ satisfying $\sum_{x\in V} \mu(x) = 1$. Suppose that two probability distributions μ_1 and μ_2 have finite support. A *coupling* between μ_1 and μ_2 is a mapping $A : V \times V \to [0, 1]$ with finite support such that

$$\sum_{y\in V} A(x, y) = \mu_1(x) \text{ and} \sum_{x\in V} A(x, y) = \mu_2(y).$$

Definition 2.2. The *transportation distance* between two probability distributions μ_1 and μ_2 is defined as follows:

$$W(\mu_1, \mu_2) = \inf_A \sum_{x,y\in V} A(x, y)d(x, y),$$

where the infimum is taken over all coupling A between μ_1 and μ_2.

By the theory of linear programming, the transportation distance is also equal to the optimal solution of its dual problem. Thus, we also have

$$W(\mu_1, \mu_2) = \sup_f \sum_{x\in V} f(x)[\mu_1(x) - \mu_2(x)]$$

where f is a Lipschitz function satisfying

$$|f(x) - f(y)| \le d(x, y).$$

Definition 2.3. [10] Let $G = (V, E)$ be a simple graph, for any $x, y \in V$ and $\alpha \in [0, 1]$, the *α-Ricci curvature* κ_α is defined to be

$$\kappa_\alpha(x, y) = 1 - \frac{W(\mu_x^\alpha, \mu_y^\alpha)}{d(x, y)},$$

where the probability distribution μ_x^α is defined as:

$$\mu_x^\alpha(z) = \begin{cases} \alpha, & \text{if } z = x, \\ \dfrac{1-\alpha}{d_x}, & \text{if } z \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4. [8] Let $G = (V, E)$ be a simple graph, for any $x, y \in V$, the Lin-Lu-Yau *Ricci curvature* $\kappa(x, y)$ is defined as

$$\kappa(x, y) = \lim_{\alpha \to 1} \frac{\kappa_\alpha(x, y)}{1 - \alpha},$$

where $\kappa_\alpha(x, y)$ is the α-Ricci curvature defined in above definition.

Naturally, a Ricci-flat graph is defined to be a graph in which the Ricci curvature of each edge is zero.

Definition 2.5. [8] A graph G is *Ricci-flat* if $\kappa(x, y) = 0$ for all edges $xy \in E$.

Next, we provide definitions for some properties of a graph that concern its symmetries, more precisely its automorphism group.

Definition 2.6. A graph G is *edge-transitive* if its automorphism group acts transitively on its edges.

Definition 2.7. A graph G is *vertex-transitive* if its automorphism group acts transitively on its vertices, i.e., for all pairs of vertices $v_1, v_2 \in V$ there exists an automorphism $\varphi : v_1 \mapsto v_2$.

Definition 2.8. A graph G is *symmetric* if it is both edge-transitive and vertex-transitive.

Definition 2.9. A graph G is *arc-transitive* (also called *symmetric* by some authors) if its automorphism group acts transitively on ordered pairs of adjacent vertices, i.e., for all ordered pairs of adjacent vertices $(u_1, v_1), (u_2, v_2)$, there exists an automorphism $\varphi : u_1 \mapsto u_2, v_1 \mapsto v_2$.

Although in general symmetric graphs are not necessarily arc-transitive, for graphs of odd degree, the two notions are equivalent. The following lemma can be proven by considering the two orbits for the arcs in a symmetric but not arc-transitive graph under the automorphism group and comparing the indegree and outdegree of an vertex in the directed graph induced by the orbits.

Lemma. *If a graph G is of odd degree, then it is arc-transitive if and only if it is symmetric.*

3 Local structures with zero curvature

The Ricci-curvature of an edge xy describes roughly the "closeness" of the neighbors of vertices x and y. In order to formulate how close the two sets of neighbors $\Gamma(x)$ and $\Gamma(y)$ are, we define the local characteristics of edge xy as follows.

Consider all possible bijective pairings $p : \Gamma(x) \setminus \{y\} \to \Gamma(y) \setminus \{x\}$ between neighbors of x and y excluding themselves such that each neighbor of x is paired uniquely with a neighbor of y. Sort all the distances between paired vertices $d(x_i, p(x_i))$ into a non-decreasing sequence $S(p)$. Let $S(p')$ be the least sequence by lexicographic order taken from all possible pairings p between the neighbor sets. The *local characteristics* (N_0, N_1, N_2) of edge xy is defined such that N_i is the number of occurrences of i in the sequence $S(p')$. In other words, N_i describes the number of $(i+3)$-cycles C_{i+3} supporting edge xy with disjoint pairs of neighbors of x and y.

The curvature of an edge in a regular graph is then completely determined by its local characteristics.

Lemma 3.1. *Let xy be an edge in a d-regular graph G with local characteristics (N_0, N_1, N_2). Then the Ricci-curvature of the edge xy is given by*

$$\kappa(x,y) = -2 + \frac{4 + 3N_0 + 2N_1 + N_2}{d}.$$

Proof. Since G is d-regular, we have $\mu_x^\alpha(x) = \mu_y^\alpha(y) = \alpha$, $\mu_x^\alpha(y) = \mu_y(x) = \frac{1-\alpha}{d}$, and $\mu_x^\alpha(v_x) = \mu_y(v_y) = \frac{1-\alpha}{d}$ for $v_x \in \Gamma(x) - \{y\}$ and $v_y \in \Gamma(y) - \{x\}$. The main idea of the proof is to show that the optimal transport plan is to transfer $\alpha - \frac{1-\alpha}{d}$ from vertex x to y, and $\frac{1-\alpha}{d}$ from vertices in $\Gamma(x) - \{y\}$ to their paired vertex in $\Gamma(y) - \{x\}$ in the distance-minimizing pairing p'.

Let $S(p')$ be the least sequence associated with the pairing p' used in the above definition of the local characteristics of edge xy . Let $A(u,v) : V \times V \to [1,0]$ be a coupling function such that

$$A(u,v) = \begin{cases} \alpha - \dfrac{1-\alpha}{d}, & \text{if } u = x,\ v = y, \\ \dfrac{1-\alpha}{d}, & \text{if } v = p'(u), \\ 0, & \text{otherwise.} \end{cases}$$

Since we'll be taking the limit as $\alpha \to 1$, assume that $\alpha > \frac{1-\alpha}{d}$. Then the transportation distance is bounded above by

$$\begin{aligned} W(\mu_x^\alpha, \mu_y^\alpha) \leq & \sum_{u,v \in V} A(u,v) d(u,v) \\ =& A(x,y)d(x,y) + \sum_{d(u,p'(u))=1,2,3} A(u,p'(u)) d(u,p'(u)) \\ =& (\alpha - \frac{1-\alpha}{d}) \cdot 1 + \frac{1-\alpha}{d} \cdot (N_1 + 2N_2 + 3(d - 1 - N_0 - N_1 - N_2)) \\ =& 3 - 2\alpha - \frac{1-\alpha}{d}(4 + 3N_0 + 2N_1 + N_2). \end{aligned}$$

In order to differentiate between the paired neighbors of x and y, define the following sets of vertices:

$$\begin{aligned} V_0 &= \{v \in \Gamma(x) - \{y\} \mid d(v, p'(v)) = 0\}, \\ X_1 &= \{v \in \Gamma(x) - \{y\} \mid d(v, p'(v)) = 1\}, \\ X_2 &= \{v \in \Gamma(x) - \{y\} \mid d(v, p'(v)) = 2\}, \\ X_3 &= \{v \in \Gamma(x) - \{y\} \mid d(v, p'(v)) = 3\}, \\ Y_3 &= \{v \in \Gamma(y) - \{x\} \mid d(p'^{-1}(v), v) = 3\}. \end{aligned}$$

We define a Lipschitz function $f : V \to \mathbb{R}$ by the following procedure:

1. $f(x) = 2$, $f(y) = 1$, $f(x_2) = 3$ for $x_3 \in X_3$, and $f(y_3) = 0$ for $y_3 \in Y_3$.

2. For $v_0 \in V_0$, if $v_0 \in \Gamma(X_3)$, then $f(v_0) = 2$; otherwise $f(v_0) = 1$. For $x_1 \in X_1$, if $x_1 \in \Gamma(X_3)$, then $f(x_1) = 2$ and $f(p'(x_1)) = 1$; otherwise $f(x_1) = 3$ and $f(p'(x_1)) = 2$. For $x_2 \in X_2$, if $x_2 \in \Gamma(X_3)$, then $f(x_2) = 2$, $f(p'(x_2)) = 0$, and $f(v_2) = 1$ for all $v_2 \in \Gamma(x_2) \cup \Gamma(p'(x_2))$; otherwise $f(x_1) = 3$, $f(p'(x_2)) = 1$, and $f(v_2) = 2$.

3. For the remaining vertices v, if $v \in \Gamma(x)$ for $f(X) = 3$, then $f(v) = 2$; otherwise $f(v) = 1$.

It is easy to check that f is indeed 1-Lipschitz, and as a result the transportation distance is bounded below by

$$
\begin{aligned}
W(\mu_x^\alpha, \mu_y^\alpha) \geq & \sum_{v\in V} f(v)[\mu_x^\alpha(v) - \mu_y^\alpha(v)] \\
=& f(x)(\alpha - \frac{1-\alpha}{d}) + f(y)(\frac{1-\alpha}{d} - \alpha) + \sum_{v\in V_0}(\frac{1-\alpha}{d} - \frac{1-\alpha}{d}) \\
&+ \sum_{v\in\Gamma(x)-\{y\}-V_0} f(v)(\frac{1-\alpha}{d} - 0) + \sum_{v\in\Gamma(y)-\{x\}-V_0} f(v)(0 - \frac{1-\alpha}{d}) \\
=& (f(x) - f(y))(\alpha - \frac{1-\alpha}{d}) + \frac{1-\alpha}{d}(\sum_{i=1}^{3}\sum_{x_i\in X_i}(f(x_i) - f(p'(x_i)))) \\
=& \frac{1-\alpha}{d} \cdot (N_1 + 2N_2 + 3(d-1-N_0-N_1-N_2)) \\
=& 3 - 2\alpha - \frac{1-\alpha}{d}(4 + 3N_0 + 2N_1 + N_2).
\end{aligned}
$$

Since the two bounds are equal, we have

$$W(\mu_x^\alpha, \mu_y^\alpha) = 3 - 2\alpha - \frac{1-\alpha}{d}(4 + 3N_0 + 2N_1 + N_2).$$

Therefore, the Ricci curvature of edge xy is

$$\kappa(x,y) = \lim_{\alpha\to 1}\frac{1 - W(\mu_x^\alpha, \mu_y^\alpha)}{1-\alpha} = -2 + \frac{4 + 3N_0 + 2N_1 + N_2}{d}.$$

□

Corollary 3.1.1. *Let xy be an edge in a Ricci-flat 5-regular graph G. Then the local characteristics (N_0, N_1, N_2) of edge xy must be one of the following five types listed in Table 3.1.*

Type-A	$(2,0,0)$
Type-B	$(1,1,1)$
Type-C	$(1,0,3)$
Type-D	$(0,3,0)$
Type-E	$(0,2,2)$

Table 3.1: Local characteristics for edges in Ricci-flat 5-regular graphs

Proof. With $\kappa = 0$ and $d_x = 5$, Lemma 3.1 gives

$$3N_0 + 2N_1 + N_2 = 6.$$

Since there are only 4 vertices in $\Gamma(x) - y$, we have $N_0 + N_1 + N_2 \leq 4$. All solutions of the above are given in Table 3.1. A schematic drawing of each local structure is shown in Fig. 3.1. Note that vertices that are not neighbors of x and y may be the same vertex as other vertices in the graph as long as the local characteristic of xy is still satisfied. □

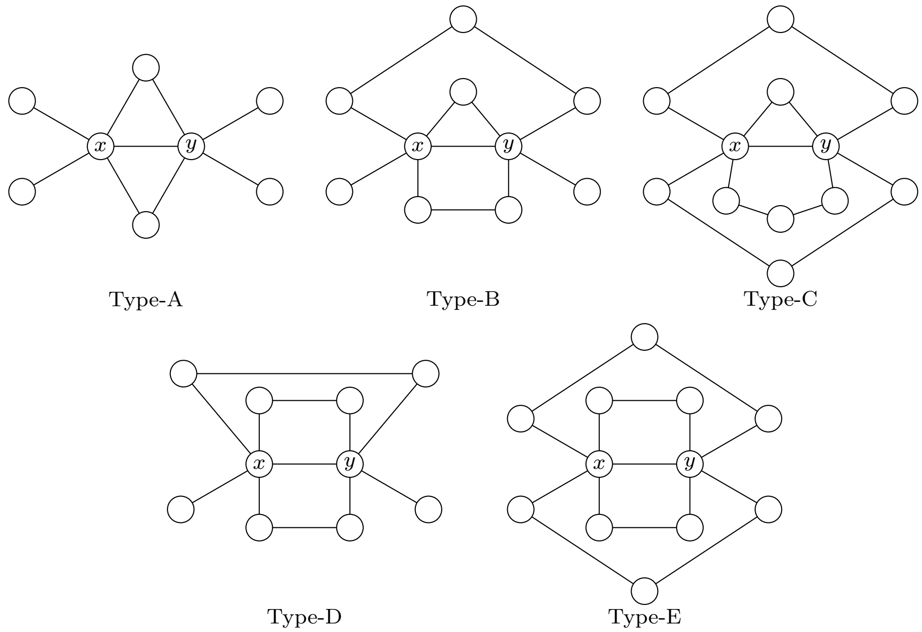


Figure 3.1: Local structures of edge xy in Ricci-flat 5-regular graphs.

It is worth noting that in each type of local structure, at least two pairs of vertices given by the pairing p' have to have distance less than 3. Moreover, excluding type-A, each type requires at least three pairs of vertices with distance less than 3.

4 Ricci-flat 5-regular symmetric graphs

In this section, we classify Ricci-flat 5-regular graphs G that are symmetric.

Theorem 4.1. *If G is a Ricci-flat 5-regular symmetric graph, then G is isomorphic to RF^5_{72}.*

For a symmetric graph G, every edge in G must have the same local structure. Therefore, we classify G based on the local structure of its edges.

4.1 Ricci-flat 5-regular symmetric graphs of girth 3

Lemma 4.2. *If G is a Ricci-flat 5-regular symmetric graph, then the edges in G are not type-A.*

Proof. Let xy be an edge in G, v_1, v_2 be common vertices of x and y, and x_1, x_2, y_1, y_2 be the neighbors of x and y respectively, as shown in Fig. 4.1. Consider edge xx_1, which needs to be in two C_3 for it to be type-A. Clearly, $x_1 \nsim y$ considering edge xy, so x_1 must be connected to two of the vertices in the set $\{v_1, v_2, x_2\}$. Since v_1 and v_2 are interchangeable, i.e., there exists an automorphism $\varphi : v_1 \mapsto v_2$, we have wlog $x_1 \sim v_1$. Next, we consider edge v_1y. Note that $d(x_1, v_2) = 2$, so we must have either $v_1 \sim v_2$ or $x_1 \sim y$. However, both option add a third C_3 on edge xv_1 or xy, and we have reached a contradiction.

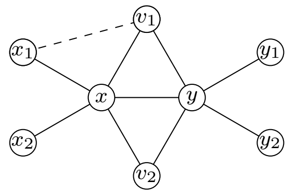


Figure 4.1

□

Lemma 4.3. *If G is a Ricci-flat 5-regular symmetric graph, then the edges in G are not type-B or type-C.*

Proof. Consider a vertex x_0 in G and its neighbors $x_i, 1 \leq i \leq 5$. Since every edge is type-B or type-C, it is in a C_3. For edge x_0x_1, wlog $x_1 \sim x_2$. For edge x_0x_3, wlog $x_3 \sim x_4$. Then, edge x_0x_5 cannot be in a C_3 since connecting x_5 with any other vertex will result in two C_3 on an edge, which is a contradiction since none of the edges are type-A.

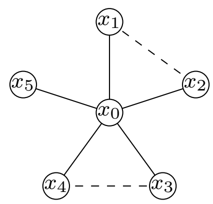


Figure 4.2

□

4.2 Ricci-flat 5-regular symmetric graphs of girth 4

Before proving that Ricci-flat 5-regular symmetric graphs with type-D edges do not exist, we prove a short lemma using the technique of double counting to show that a Ricci-flat graph containing only type-D edges must contain two 4-cycles sharing two edges, that is, the subgraph shown in Fig. 4.3 which is isomorphic to the complete bipartite graph $K_{3,2}$.

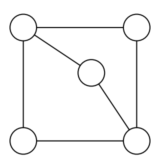


Figure 4.3

Lemma 4.4. *If G is a Ricci-flat 5-regular graph containing only type-D edges, then it contains $K_{3,2}$ as a subgraph.*

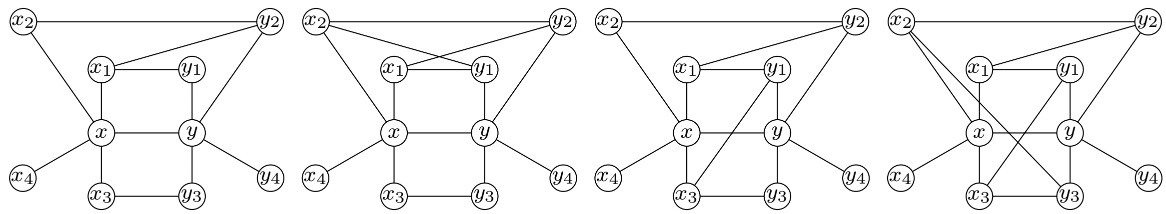
Proof. We show by contradiction that there doesn't exist a Ricci-flat 5-regular graph with only type-D edges that does not contain $K_{2,3}$, i.e., in which all 4-cycles share at most one edge. Suppose such a graph G exists. Consider a vertex x_0 in G and its neighbors $x_i, 1 \leq i \leq 5$. Since all C_4 share at most one edge, each one of the five edges x_0x_i is in exactly three C_4. Thus, the number of ordered pair (x_0x_i, C_4^*) where $x_0x_i \in C_4^*$ should be 15. On the other hand, each C_4 through vertex x_0 contains two edges x_i and x_ix_j. Thus, the number of ordered pairs (x_0x_i, C_4^*) should be even, and we have reached a contradiction. □

Lemma 4.5. *If G is a Ricci-flat 5-regular symmetric graph, then the edges in G are not type-D.*

Proof. Since G is symmetric and of odd degree, it must be arc-transitive. As a result, the neighborhood of an edge $u_1v_1 \in G$ denoted by $\Gamma(u_1v_1)$, i.e., the subgraph induced by $\Gamma(u_1) \cup \Gamma(v_1)$ must be isomorphic to the neighborhood of any other edge $\Gamma(u_2v_2)$. Since by Lemma 4.4, G must contain $K_{3,2}$ as a subgraph, each edge is in a $K_{3,2}$ as a result of Lemma 4.4. We classify all possible neighborhoods of an edge xy such that xy is in a $K_{3,2}$ and there is an automorphism $\varphi : \Gamma(xy) \to \Gamma(xy)$ mapping xy to yx. Let x_i and y_i be the neighbors of x and y excluding themselves, and wlog $x_i \sim y_i$ for $i = 1, 2, 3$ and $d(x_4, y_4) = 3$. In order to form a $K_{3,2}$ on xy, we have wlog either $x_1 \sim y_2$ or $x_1 \sim y_4$.

Assume that $x_1 \sim x_2$ and $x_1 \nsim y_4$, we break into two cases based on the number of connections between x_i and y_j.

1. Suppose each x_i, $i = 1, 2, 3$ is connected to at most one $y_j, j \neq i$.



(a) (b) (c) (d)

Figure 4.4

   (a) Assume that xy has the neighborhood shown in Fig. 4.4(a). Consider edge xx_4, which cannot form a C_4 through xy as it has distance 3 to all the non-adjacent vertices. Thus, it must form a C_4 with each x_i, $i = 1, 2, 3$ by connecting x_4 to a new neighbor of x_i namely z_i. For the neighborhood of xx_4, we need to connect one of z_i to x_j, $i, j \in \{1, 2, 3\}$. Note that xx_1 is already in three C_4, namely x_1y_1yx, $x_1y_2x_2x$ and $x_1z_1x_4x$. Since we have $y_2 \sim y$, its neighborhood including the fifth neighbor of x_1 is isomorphic to $\Gamma(xy)$. Thus, the neighbors of x_1 and x are not further connected, and we have $x_1 \nsim z_2, z_3$ and $z_1 \nsim x_2, x_3$. Therefore, we must have either $x_2 \sim z_3$ or $z_2 \sim x_3$.

   If $x_2 \sim z_3$ as in Fig. 4.5(a), consider edge xx_2, which is already in three C_4. Let v be the fifth neighbor of x_2, we have $d(v, x_1) = 3$. However, as x_1 is connected to y_2, a neighbor of x_2, the neighborhood of xx_2 is not isomorphic to $\Gamma(xy)$, contradiction.

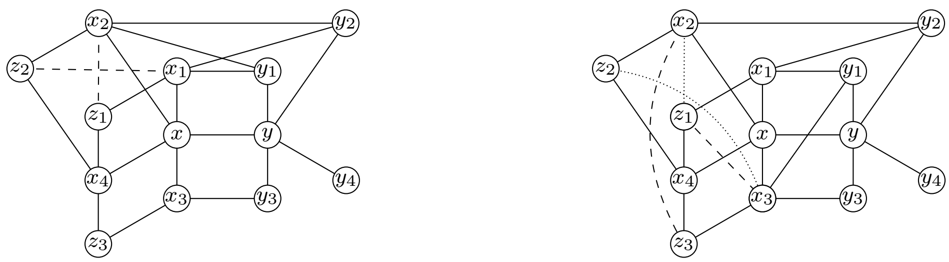
   If $z_2 \sim x_3$ as in Fig. 4.5(b), then edge xx_3 is in three C_4 and has isomorphic neighborhood to xy. Consider edge xx_2, which needs another C_4 formed through a new neighbor of x_2 namely v since x_2 cannot connect to any of the existing vertices. However, $v \nsim x_1$ considering the neighborhood of xx_1, $v \nsim x_3$ considering the neighborhood of xx_3. Thus, the third C_4 on edge xx_2 cannot be formed, a contradiction.

(a) (b)

Figure 4.5

(b) Assume that xy has the neighborhood shown in Fig. 4.4(b). Similar to Case 1(a), we have $z_i \sim x_i$ for $i = 1, 2, 3$ where z_i are neighbors of x_4 as in Fig. 4.6(a). Since the neighborhood of edge xx_4 needs to be isomorphic to $\Gamma(xy)$, we must have wlog either $z_1 \sim x_2$ or $z_1 \sim x_3$. However, $d(z_1, x_3) = 3$ considering edge xx_1, which is already in three C_4, so we must have $z_1 \sim x_2$ and also $z_2 \sim x_1$. Next, we consider edge xx_3, which is in two C_4 and needs to form a C_4 through either xx_1 or xx_2. Since xx_1 and xx_2 are equivalent edges under an automorphism, let the C_4 pass through xx_1. Since x_1 is at maximum degree, x_3 must be connected to one of the neighbors of x_1. However, none of the neighbors of x_1 can be connected to x_3 given the neighborhood structure of edges xy and xx_4, and we have reached a contradiction.

(c) Assume that xy has the neighborhood shown in Fig. 4.4(c). Similar to Case 1(a), we have $z_i \sim x_i$ for $i = 1, 2, 3$ where z_i are neighbors of x_4 as in Fig. 4.6(b), and we need to connect neighbors of x_4 and x so that the neighborhood of xx_4 is isomorphic to $\Gamma(xy)$. Since xx_1 is already in three C_4, we have $x_1 \nsim z_2, z_3$. Thus, we have either $x_2 \sim z_3$ and $x_3 \sim z_1$ (shown with dashed lines), or $x_2 \sim z_1$ and $x_3 \sim z_2$ (shown with dotted lines). However, in each case four C_4 are created on edge xx_1 and xx_3 respectively, a contradiction.

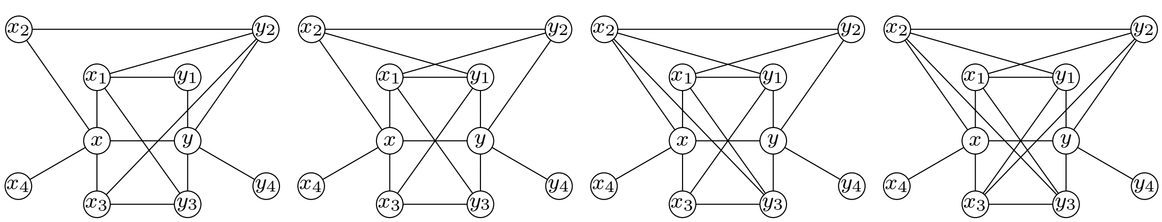


(a) (b)

Figure 4.6

(d) Assume that xy has the neighborhood shown in Fig. 4.4(d) respectively. The argument for Case 1(c) applies similarly.

2. Suppose each x_i, $i = 1, 2, 3$ is connected to at most two $y_j, j \neq i$. Wlog, let $x_1 \sim y_2, y_3$, and by the automorphism φ there needs to be a vertex y_j such that it is connected to two $x_i, i \neq j$. By casework, we have the following potential neighborhoods of xy shown in Fig. 4.7 in which there exists an automorphism φ.



(a) (b) (c) (d)

Figure 4.7

(a) Assume that xy has the neighborhood shown in Fig. 4.7(b). We relabel the vertices by interchanging y_1 and y_2 and redraw the graph in Fig. 4.8 to highlight the symmetry and its similarity to the following cases.

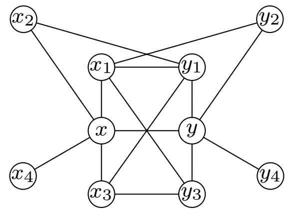


Figure 4.8

Consider edge xx_4 and note that x_4 has distance 3 to y_4 and the fifth neighbor of x_1. Since it does not connect to any existing vertex either, it cannot form a C_4 through both xx_1 and xy, a contradiction.

(b–d) Assume that xy has the neighborhood shown in Fig. 4.7(b)-(d) respectively. Consider edge xx_4 and a similar contradiction arises as in Case 2(a).

Therefore, we proved that we must have wlog $x_1 \sim y_4$. Then, x_4 must also be connected to a neighbor of y given the automorphism φ. Note that $x_4 \nsim y_1$ or else $d(x_4, y_4) = 1$, so we have wlog $x_4 \sim y_3$. Moreover, note that $x_4 \nsim y_2$, since if X_4 is connected to two neighbors of y, y_4 must be connected two neighbors of x as well, resulting in $d(x_4, y_4) = 1$. Similarly, $y_4 \nsim x_2, x_3$. Therefore, we have $x_1 \sim y_4$ and $x_4 \sim y_3$. We combine this with the discussion of whether x_i and y_j where $i, j \in \{1, 2, 3\}, i \neq j$ is connected above, and arrive at several possibility for $\Gamma(xy)$.

1. Suppose $x_i \nsim y_j$ for all $i, j \in \{1, 2, 3\}$ and $i \neq j$ as in Fig. 4.9.

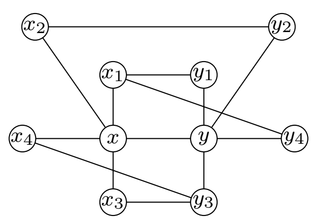


Figure 4.9

Consider edge xx_1. Let z_1, z_2 be the two new neighbors of x_1. Then, for $\Gamma(xx_1)$ to be isomorphic to $\Gamma(xy)$, wlog we have z_1 connected to two neighbors of x excluding x_1, and z_2 connected to the remaining neighbor of x. Since x_3 and x_4 are interchangeable, there are two cases.

(a) Assume wlog that $z_1 \sim x_2, x_4$ and $z_2 \sim x_3$. Consider edge xx_4. Note that it cannot have an isomorphic neighborhood to xy because y_3 and z_1, two neighbors of x_4, have degree 3 in the neighborhood of xx_4, a contradiction.

(b) We have that $z_1 \sim x_3, x_4$ and $z_2 \sim x_2$. Consider edge z_1x_4, which is in two C_4, namely $z_1x_1xx_4$ and $z_1x_3y_3x_4$. However, we also have xx_3, which makes it impossible for $\Gamma(z_1x_4)$ to be isomorphic to $\Gamma(xy)$, a contradiction.

2. Suppose there exists x_i, $i \in \{1, 2, 3\}$ such that $x_i \sim y_j$ for some $j \in \{1, 2, 3\} - \{i\}$, then there are only two non-isomorphic possibilities for $\Gamma(xy)$ by noticing that the automorphism φ sending xy to yx must be $\varphi : x_1 \mapsto y_3, x_2 \mapsto y_2, x_3 \mapsto y_1, x_4 \mapsto y_4$.

(a) (b)

Figure 4.10

(a) Assume that xy has the neighborhood shown in Fig. 4.10(a). Let the fifth neighbor of x_1 be z_1 and the fifth neighbor of y_3 be z_2. Consider edge x_1y_3, which needs one more C_4 and it must be formed by connecting $z_1 \sim z_2$. This way, edges xx_1 and yy_3 have isomorphic neighborhoods to xy. Next, we consider edge xx_1, which is already in three C_4. To make $\Gamma(xx_1)$ isomorphic to $\Gamma(xy)$, we must have $x_2 \sim z_1$.

Next, we consider edge x_4x. Notice that y_3, a neighbor of x_4, is connected to four neighbors of x, thus it must be mapped to x_1 by the automorphism sending edge x_4x to xy. Thus, x_2, the only neighbor of x not connected to y_3, must be connected to a new neighbor of x_4 namely w_1. Similarly, since vertices x_3 and x_4 are interchangeable, the same analysis applies and x_2 must be connected to a neighbor of x_4. Note that $x_4 \nsim w_1$ since if so, w_1 as a neighbor of x_3 would be connected to two neighbors of x, resulting in $\Gamma(xx_3)$ no longer possible to be isomorphic to $\Gamma(xy)$. Thus, we must have $x_2 \sim w_2 \sim x_4$. However, in this case, xx_2 would be in four C_4, a contradiction.

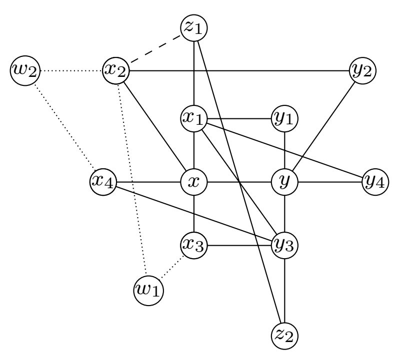


Figure 4.11

(b) Assume that xy has the neighborhood shown in Fig. 4.10(b). Let the fifth neighbor of x_1 be z. Consider edge xx_1, which is in two C_4, so another C_4 needs to be formed through x_1z. Since x_3 and x_4 are interchangeable, let $z \sim x_4$. To make the $\Gamma(xx_1)$ isomorphic to $\Gamma(xy)$, we must have $z \sim x_2, x_3$. However, in this way, $\Gamma(xx_2)$ cannot be isomorphic to $\Gamma(xy)$, a contradiction.

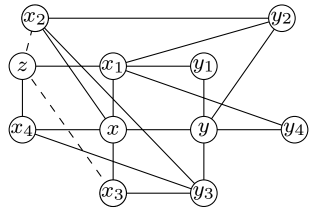


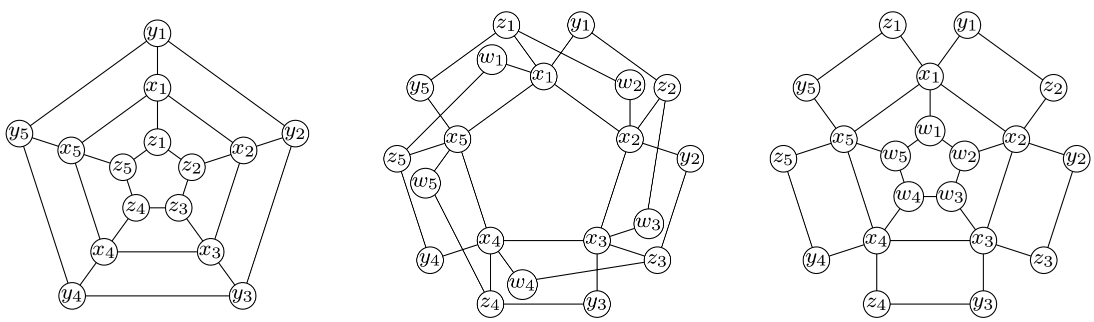
Figure 4.12

□

Next, we move onto symmetric graphs with type-E edges.

Lemma 4.6. *If G is a Ricci-flat 5-regular symmetric graph with type-E edges, then it is isomorphic to RF^5_{72}.*

Proof. We start by considering a C_5 in G and denote its vertices $x_i, 1 \leq i \leq 5$. Since each edge is type-E, it needs to be supported on two C_4. There are only three arrangements of the C_4 on edges in the C_5 under consideration such that each arc in the C_5 are in the same orbit under the automorphism group of this subgraph, since for the two C_4 on an edge $x_i x_{i+1}$, at least one of them is adjacent to a C_4 on the neighboring edge $x_{i+1}xi + 2$. If both C_4 on an edge are adjacent to the two C_4 on neighboring edges of the C_5, we have the first case in Fig. 4.13(a). When only one C_4 is adjacent to a C_4 on the neighboring edges, if there are no three adjacent C_4 in a row, that is, adjacent C_4 on edges $x_i x_{i+1}, x_{i+1} x_{i+2}, x_{i+2} x_{i+3}$, we have the second case shown in Fig. 4.13(b); otherwise, we have the third case shown in Fig. 4.13(c).



(a) (b) (c)

Figure 4.13

Since G is symmetric, all the C_5 in G must have a local structure that is isomorphic to the subgraph shown above in each case. We construct the graph with the aid of a curvature calculator [5]. In the first two cases, contradiction arises in the construction process, while the local structure of a C_5 in the third case can be successfully expanded into a Ricci-flat 5-regular graph, which is isomorphic to what we denote as RF^5_{72}. The 2-neighborhood and 3-neighborhood of a vertex in RF^5_{72} are shown in Fig. 1.3. □

This concludes our proof of Theorem 4.1.

5 General Ricci-flat 5-regular graphs of girth 3

In this section, we prove some lemmas that show the nonexistence of certain subgraphs of girth 3 in a Ricci-flat 5-regular graph.

We start by considering graphs containing adjacent triangles, that is, an edge of type-A.

Lemma 5.1. *If G is a Ricci-flat 5-regular graph, then it does not contain subgraph H shown in Fig. 5.1.*

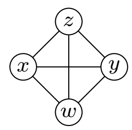


Figure 5.1

Proof. Suppose G contains H, let x_1, y_1, z_1, w_1 each be another neighbor of x, y, z, w respectively. Consider the edge xx_1 and let x_2 be another neighbor of x_1, which has to be distinct from all existing vertices. Note that $d(x_2, y) = d(x_2, z) = d(x_2, w) = 3$ because xy, xz, xw have to be Type-A and $d(x_1, y_1) = d(x_1, z_1) = d(x_1, w_1) = 3$. Therefore, edge xx_1 cannot be one of the types since any type requires at least two pairs of vertices with distance less than 3, which is a contradiction. $\square$

Lemma 5.2. *If G is a Ricci-flat 5-regular graph, then it does not contain subgraph H shown in Fig. 5.2.*

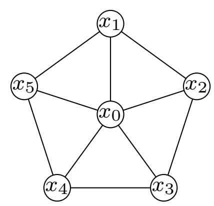


Figure 5.2

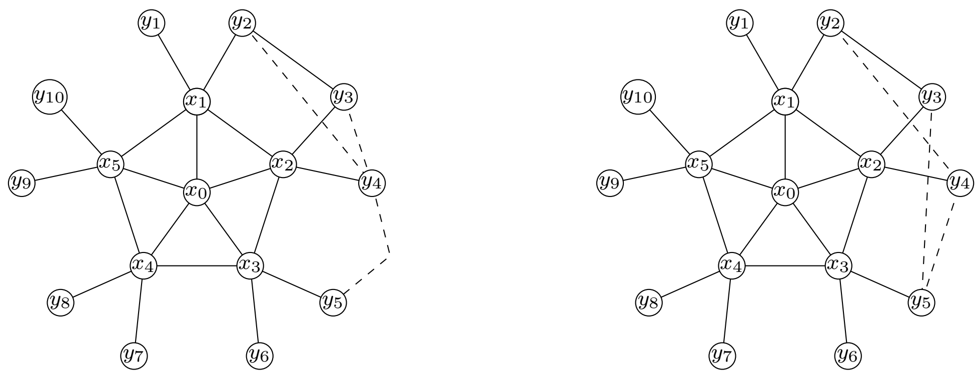
Proof. Suppose G contains H, we consider the remaining two neighbors of x_1, which must be two new vertices y_1, y_2 since $x_1 \nsim x_3, x_4, x_5$ by Lemma 5.1. Similarly, x_2 needs two new vertices as $x_2 \nsim y_1, y_2$, or otherwise edge x_1x_2 is already supported on one C_3 as $d(x_0, x_0) = 0$ and one C_5 as $d(x_3, x_5) = 2$. We continue the same line of reasoning for x_3, x_4, x_5 and let the new neighbors for x_i be y_{2i-1} and y_{2i}. Consider edge x_1x_2, which has to be either type-B or type-C.

1. x_1x_2 is type-B, then it is supported on a C_4 through new neighbors of x_1 and x_2, wlog we have $y_2 \sim y_3$ and $d(y_1, y_4) = 3$ as shown in Fig. 5.3. Next, we consider edge x_2x_3, which can be either type-B or type-C.

   (a) If x_2x_3 is type-C as in Fig. 5.3(a), we have wlog $d(y_4, y_5) = 2$ since y_5 and y_6 are equivalent. Consider edge x_2y_4 which is already in one C_5. It must be type-B since it cannot form any C_3, C_4, C_5 through x_2x_0 without forming a C_4 through x_2x_1. It needs to form a C_3 and a C_4 through x_2x_1 and x_2x_3. Since $x_1 \nsim y_4$ considering edge x_1x_2, we have $y_3 \sim y_4$ and $y_2 \sim y_4$ and as a result $d(y_1, y_3) = d(y_1, y_4) = 3$ considering edge x_1x_2. Next, we consider edge x_1y_1, since $d(y_1, x_3) = 3$ by type-A of edge x_0x_1, then it cannot form any C_3, C_4, C_5 through edge x_1x_2, and it cannot form a C_4 through x_1x_0. Thus it is not type-C, D, E. To be type-B, it needs $y_1 \sim y_2$ which contradicts to $d(y_1, y_3) = 3$. Hence, x_1y_1 cannot be any of the good types. A contradiction.

   (b) Therefore, edge x_2x_3 is type-B and forms a C_4 with either x_2y_3 or x_2y_4.

i. If a C_4 is formed through x_2y_3, we have wlog $y_3 \sim y_5$ as in Fig. 5.3(b). Consider edge x_2y_4, which is clearly not type-A. It cannot be in a C_3, otherwise we have $y_3 \sim y_4$ which leads to x_2y_3 being in two C_4 and one C_3. Since $d(y_1, y_4) = d(y_6, y_4) = 3$, edge x_2y_4 cannot be type-E as a C_5 cannot be formed through x_2x_0 without forming C_4. Thus iAt is type-D and $y_4 \sim y_2, y_5$. We have $d(y_1, y_3) = d(y_1, y_4) = 3$ considering edge x_1x_2. Then, we reach a contradiction that x_1y_1 is not any of the types by the same argument as in the above case.



(a) (b)

Figure 5.3

ii. Therefore, we must have $y_4 \sim y_5$. Applying the above argument to edge x_3x_4, we see that $y_6 \sim y_7$ or $y_6 \sim y_8$ by rotational symmetry.
Continuing this way we obtain the structure shown in Fig. 5.4. Consider the new neighbors of y_4. We claim that y_4 cannot have three distinct new neighbors. If so, let the new neighbors be v_1, v_2, v_3. Clearly $d(x_0, v_i) = 3, d(x_1, v_i) \geq 2$ and thus edge x_2y_4 must be type-B since it is already in one C_4 and cannot form any C_3, C_4, C_5 through x_2x_0. In this way, x_2y_4 must be in a C_3, contradicting with the assumption that y_4 has three new neighbors. As a result, y_4 must be connected to an existing vertex. We consider the possible neighbors of y_4. Note that $y_4 \nsim y_1, y_2$ considering x_1x_2, $y_4 \nsim y_7, y_8$ considering x_0x_2, and $y_4 \nsim y_9, y_{10}$ considering x_0x_5. Moreover, $y_4 \nsim y_2, y_6$ by the argument in Case (b)i.
Thus, we must have $y_4 \sim y_3$, that is, edge x_2y_4 is type-B and needs a C_5. The only way to form a C_5 supporting x_2y_4 is through x_2x_1, and we have $y_4 \sim t_1 \sim y_2$. Similarly, x_2y_3 must be type-B and needs a C_5 through x_2x_3. The C_5 cannot go through vertex t_1, since if $y_3 \sim t_1 \sim y_5$, edge y_3y_4 would be in two C_3 and one C_5. Thus, we must have $y_3 \sim t_2 \sim y_5$. However, edge y_3y_4 is now in one C_3 and two C_4, a contradiction. Therefore, we have $y_4 \nsim y_3$.

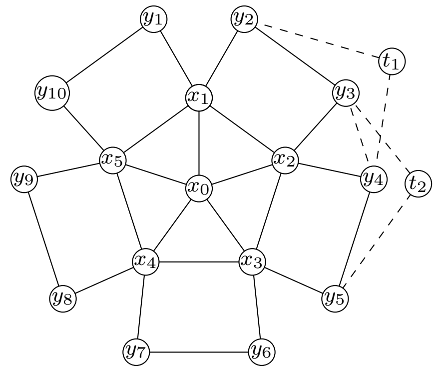


Figure 5.4

2. x_1x_2 is type-C, then it has to be supported on two more C_5 as shown in Fig. 5.5. We have wlog $d(y_1, y_3) = d(y_2, y_4) = 2$. Consider edge x_2y_3, which is in one C_5 through x_2x_1 and cannot form any C_3, C_4, C_5 through the edge x_2x_0 without forming a C_4 through x_2x_3 or through x_2x_1. Thus, edge x_2y_3 must be type-B and forms either a C_3 or a C_4 through edge x_2x_3. Since no C_3 can be formed through x_2x_3, it has to be a C_4 through x_2x_3, resulting in $y_4 \sim y_5$. In this way, edge x_2x_3 is type-B, and leads to a contradiction by similar analysis in Case 1 as x_2x_3 and x_1x_2 are equivalent under rotational symmetry. We have finished the proof that a Ricci-flat 5-regular graph does not contain subgraph H.

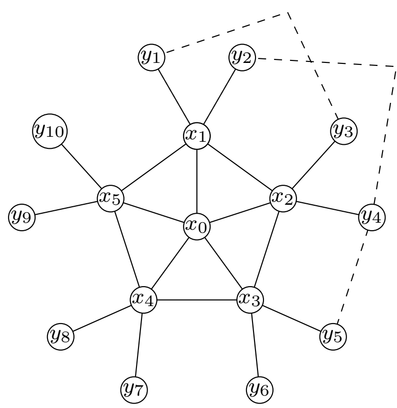


Figure 5.5

□

Lemma 5.3. *If G is a Ricci-flat 5-regular graph, then it does not contain subgraph H shown in Fig. 5.6.*

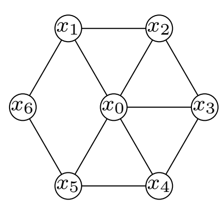


Figure 5.6

Proof. Consider edge x_0x_5, which must be type-B and needs a C_5 since it is already in a C_3 and a C_4 given that $x_1 \nsim x_5$ by Lemma 5.2.

1. Suppose it forms a C_5 through x_0x_2, then we must have $x_1 \sim t_1 \sim x_5$ shown in Fig. 5.7 as the C_5 cannot go through any new neighbors of x_2 considering that edge x_0x_2 is type-A. Let t_2 be the fifth neighbor of x_5, we have $d(t_2, x_3) = 3$ as edge x_0x_5 is type-B. Observe that vertex x_3 needs two new vertices w_1, w_2 as its neighbors. We have $t_1, x_6, x_2, x_4 \nsim w_1, w_2$ considering that edge x_0x_3 is type-A. In addition, $x_6 \nsim x_2$ otherwise there would be two C_4 supported on edge x_0x_5, similarly, $x_6 \nsim x_4$ and $t_1 \nsim x_2, x_4$. Thus $d(x_6, x_3) = d(t_1, 3) = 3$, and edge x_4x_5 cannot be type-C as there is no way to form three C_5 and thus it has to be type-B that needs a C_4.

   Vertex x_4 needs two new vertices as its neighbors u_1, u_2. Note that $x_6, t_1 \nsim u_1, u_2$ considering that edge x_0x_4 is type-A. To form the C_4 on x_4x_5, wlog let $x_5 \sim t_2 \sim u_1$. Consider edge x_4u_1, which is in a C_4. Thus, it is either type-B, type-D, or type-E.

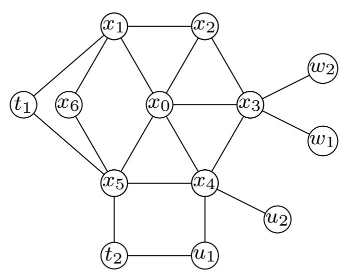
   

   Figure 5.7

   (a) If edge u_1x_4 is type-E, then there must be a C_5 passing through edge x_4x_0. The only way to form such C_5 is by connecting $u_1 \sim w_1$ and $u_1 \sim w_2$ such that a C_4 is formed through one of them with edge x_4x_3 and a C_5 is formed with the other through x_4. Now we consider the edge u_2x_4. Observe that it cannot form any C_3 or C_4 with any neighbor of x_4, thus it is not any type. A contradiction. Similarly, edge u_2x_4 is not type-E.

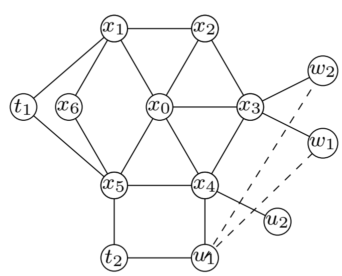
   

   Figure 5.8

   (b) If edge u_1x_4 is type-D, then $u_1 \sim w_1$ and edge x_3x_4 is type-B, we have $d(u_2, w_2) = 3$. u_2x_4 is not in any C_3 so it is also type-D. To form the third C_4 for edge u_1x_4, let u_3 be the shared neighbor of u_1, u_2. For edge u_2x_4 to be type-D, it needs two C_4 through edges x_4x_5 and x_4x_3, and it must be $u_2 \sim t_2$ and $u_2 \sim w_1$, otherwise there would be two C_4 for edge x_4x_5 or edge x_4x_3, a contradiction since both edges are type-B. Now we have $d(u_1, w_2) = d(u_2, w_2) = 3$ for edge x_3x_4. Consider edge w_2x_3, and note that $w_2 \nsim x_2$. If a C_4 is not formed through $w_2x_3x_2$, then any new neighbor of w_2 has distance 3 to vertex x_4 and vertex x_0, which results in edge w_2x_3 not being one of the five types, a contradiction. Thus $w_2 \sim w_3 \sim x_2$ and edge w_2x_3 is type-B, let $x_2 \sim w_4$, we have $d(w_1, w_4) = 3$. Observe that $w_1 \nsim w_2$ as $d(u_2, w_2) = 3$, and thus both w_1x_3 and w_2x_3 are not in any C_3. Note there cannot be a C_5 on w_1x_3 through $x_3x_0x_2$ as $d(w_1, w_4) = 3$. Then w_1x_3 is type-D whose second C_4 is formed by $w_1 \sim w_3$ and the third C_4 is formed by $w_1 \sim p_1 \sim w_2$ where p_1 is a new vertex. Now the edge w_2x_3 is in two C_4 and cannot form C_3, C_4, C_5 through edge x_3x_4 and cannot form C_3, C_4 through edge x_3x_2 so that w_2x_3 is not type-D or type-E. A contradiction.

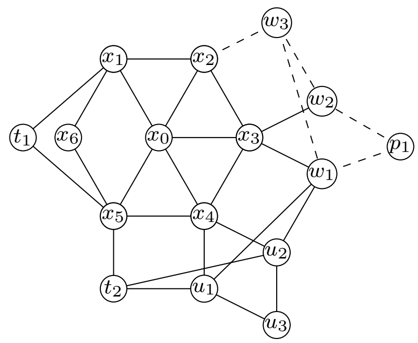
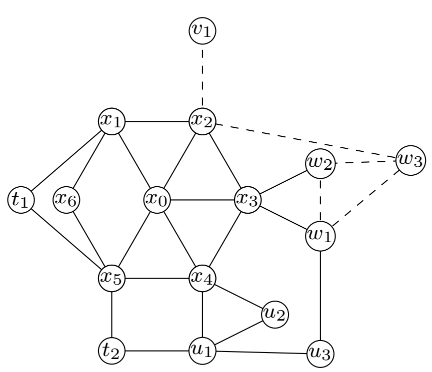
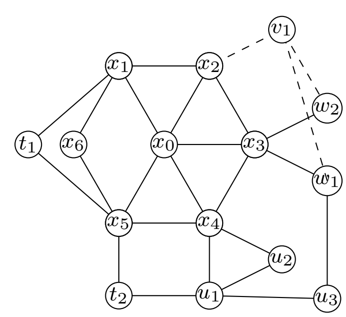


Figure 5.9

(c) Thus edge u_1x_4 is type-B and $u_2 \sim u_1$. Observe that the C_5 for edge u_1x_4 does not pass through edge x_4x_0, and as a result it must pass through edge x_4x_3 through w_1 or w_2. Wlog, let $w_1 \sim u_3 \sim u_1$. Note that $u_1 \nsim w_1, w_2$ considering edge edge u_1x_4 and $u_2 \nsim w_2$ considering edge x_3x_4. Consider edge w_1x_3, which cannot form C_4 or C_5 through x_3x_0, implying that it is not type-C or type-D. Assuming it is type-B as shown in Fig. 5.10(a), then $w_1 \sim w_2$, and it needs a C_4 which must pass through edge x_3x_2. Since $d(x_2, u_1) = 3$ considering edge x_0x_4, this C_4 cannot be obtained by $u_3 \sim x_2$. Let $x_2 \sim w_3 \sim w_1$. Similar analysis applies to edge w_2x_3 and it has to be type-B, needing a C_4 through x_2x_3. To avoid two separate C_4 on edge x_3x_2, we have $w_2 \sim w_3$. Let v_1 be the fifth neighbor of x_2 and consider edge v_1x_2. As $d(v_1, w_1) = d(v_2, w_2) = 3$ considering edge x_3x_2, $d(v_1, x_4) = d(v_1, x_5) = 3$ considering edge x_0x_2, and thus edge x_2v_1 cannot be any type. A contradiction. Therefore, edge w_1x_3 and w_2x_3 are both type-E as shown in Fig. 5.10(b) and both need a C_4 through edge x_3x_2. Without forming two C_4 supported on x_3x_2, let $w_1, w_2 \sim v_1 \sim x_2$. Now consider the C_5 passing through x_3x_0 for type-E of edge w_1x_3, but it cannot be formed without passing through x_0x_2 which leads to two distinct C_4 on edge x_3x_2. A contradiction.



(a)



(b)

Figure 5.10

2. Thus, x_0x_5 must form a C_5 through x_0x_3, and in this case we have $x_4 \sim y_1 \sim x_5$ shown in Fig. 5.11. Let y_2 be the fifth neighbor of x_4 as x_4 cannot be connected to any existing vertices. Consider edge x_3x_4, which must be either type-B or type-C.

   We claim that x_3x_4 cannot be type-C. If so, since $d(x_5, w_1) = d(x_5, w_2) = 3$, x_4y_1 and x_4y_2 must each be in a C_5 through x_4x_3. Note $y_1 \nsim y_2$ as there is C_5 passing through $y_1x_4x_3$. There is no C_3, C_4, C_5 passing through $y_2x_4x_0$ because of type-A of edge x_4x_0 and edge x_0x_2. Thus x_4y_2 cannot be any good type. A contradiction.

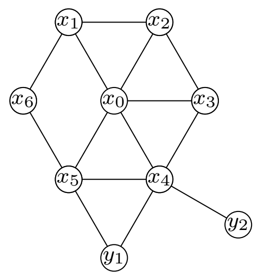
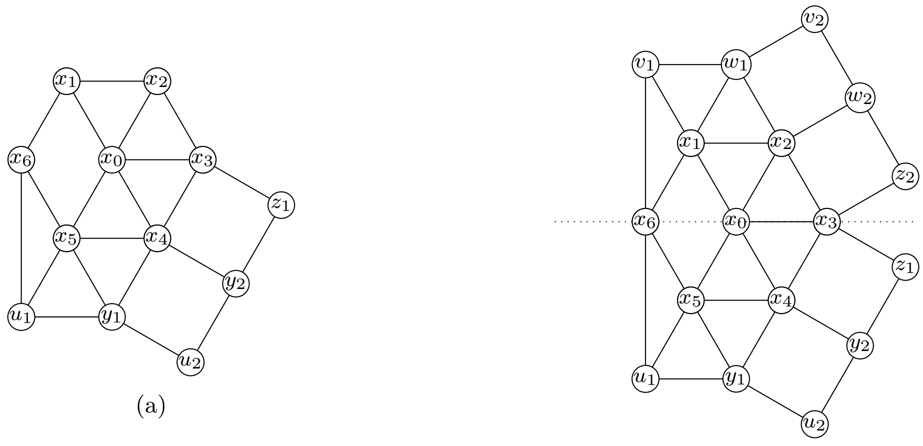


Figure 5.11

   Therefore, x_3x_4 must be type-B. Let z_1 be a new neighbor of x_3, then we have wlog $z_1 \sim y_2$ for the C_4 on x_3x_4. Note that $y_1 \nsim x_6$ because of type-A of edge x_0x_4. Thus, y_1 must have two new vertices u_1, u_2.

   Next, we consider edge x_4y_1, since $d(x_5, x_1) = 3$, we have $y_1 \nsim z_1$, thus x_4y_1 cannot form any C_3, C_4, C_5 through x_4x_3 and thus must be type-B. Since a C_4 cannot be formed through x_4x_0, we have have wlog $u_2 \sim y_2$. Then a C_5 must be formed through x_4x_0 followed by x_4x_5. Thus, we have wlog $u_1 \sim x_5$ for the C_5. See 5.12(b). Now edge x_5y_1 is also type-A, we have that the edge x_6x_5 cannot form any new C_3, C_4, C_5 through x_5x_4, thus it is not type-C or type-E. It cannot be type-D because it cannot form a C_4 through x_5y_1. Thus, x_6x_5 must be type-B and forms a C_3 by connecting x_6 and u_1.

   Next, we observe the inflectional symmetry across the dotted line with the starting configuration vertices x_i, and apply our analysis above to the upper half of the graph in Fig. 5.12(b). For the addition of new vertices, note that $z_1 \neq z_2$ as otherwise the fifth edge on vertex x_3 cannot form any C_3, C_4, C_5 through x_3x_0 and x_4x_4, and that $v_2 \neq u_2$ as otherwise edge w_1v_2 cannot be one of the five types.



(a) (b)

Figure 5.12

Finally, we observe the rotational symmetry around the 4-cycle $x_1x_0x_5x_6$, and the vertex set $\{x_5, x_1, x_0, x_4, y_1, u_1, x_6\}$ and the edges between them are isomorphic to the starting configuration of the lemma. Similarly analyzing the addition of new vertices, we obtain the graph shown in Fig. 5.13 with vertices re-labeled to highlight its symmetry. Consider edge y_1w_1, which is already in two C_4. Note that it is not possible to form another C_4 through y_1x_1 or y_2x_2. Thus, edge y_1w_1 must be type-E, note $w_1 \nsim w_4$ as edge x_1y_1 is type-A, thus we have $w_1 \sim u_3, u_8$ for the two C_5 through y_1x_1 and y_2x_2. Similarly for all y_iw_i, we have $w_2 \sim u_2, u_5$, $w_3 \sim u_4, u_7$ and $w_4 \sim u_6, u_1$.

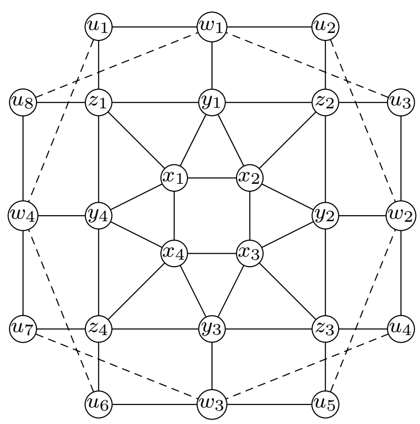


Figure 5.13

Next, we consider edge z_1u_1, which is in two C_4 through z_1y_1 and z_1y_4. Since a C_5 cannot be formed through z_1x_1, z_1u_1 must be type-D and forms a C_4 through z_1u_8. Since u_1 cannot connect to any of the existing vertices, it must have two new neighbors v_1, t_1. Thus, for the C_4 through z_1u_8, we have $u_1 \sim v_1 \sim u_8$, and edge z_1u_8 is also set as type-D. We apply the same analysis to other edges z_iu_j, and denote the new vertex that forms the C_4 as v_i. We show that all v_i are distinct vertices by eliminating the following cases in which $v_1 = v_2$ and $v_3 = v_4$.

(a) Suppose $v_1 = v_2$ as shown in Fig. 5.14. Note that $t_1 \neq t_2$ since u_1w_1 and u_2w_1 are type-D edges as they are already in three C_4. Consider edge v_1u_1, which is already in two C_4

through $u_1w_4u_8v_1$ and $u_1w_1u_2v_1$. Since no C_5 can be formed through u_1z_1 as all other neighbors of z_1 are at maximum degree, v_1u_1 is type-D and forms another C_4 through u_1t_1. Thus, we have $t_1 \sim t_3 \sim v_1$. Applying the same argument to v_1u_2, we have $t_3 \sim t_2$. Next, we consider edge v_1u_8, which is in two C_4 through $v_1u_1w_4u_8$ and $v_1u_2w_1u_8$. It cannot be type-E since no C_5 can be formed through u_8z_1 since all other neighbors of z_1 are at maximum degree, so v_1u_8 must be type-D and forms another C_4 through a new neighbor of u_8 namely t_4 and we have $t_4 \sim t_3$. Since $u_3 \nsim t_4$, let the fifth neighbor of u_3 be t_5. The same argument for v_1u_8 applies to v_1u_3, and we have $t_5 \sim t_3$. However, now edge t_3v_1 is in four C_4, contradiction.

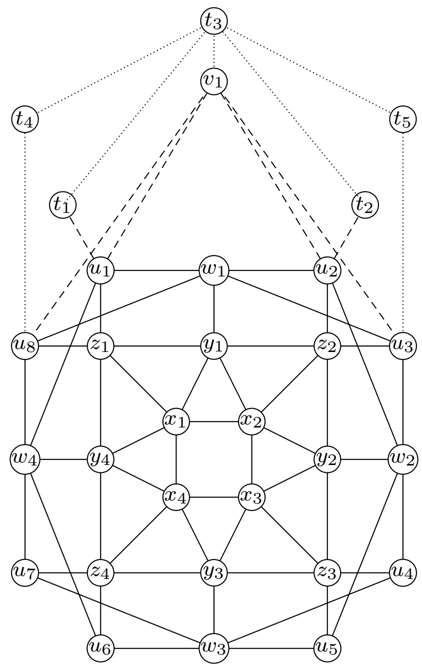


Figure 5.14

(b) Suppose $v_1 = v_3$ as shown in Fig. 5.15. Consider edge v_1u_4, which is in one C_4 through v_1u_5. Consider edge v_1u_1. If v_1 and u_1 each has a new vertex as its fifth neighbor, then by process of elimination, u_1v_1 cannot be any type. Wlog, we let $u_1 \sim u_4$ and $v_1 \sim t_1 \sim u_8$ so that we have edge v_1u_1 as type-B. Next, we consider edge u_1w_1, in which both vertices have degree 5 already and it is clear u_1v_1 cannot be any type with curvature zero. Similar contradiction arises after satisfying the requirements for edge v_1u_1.

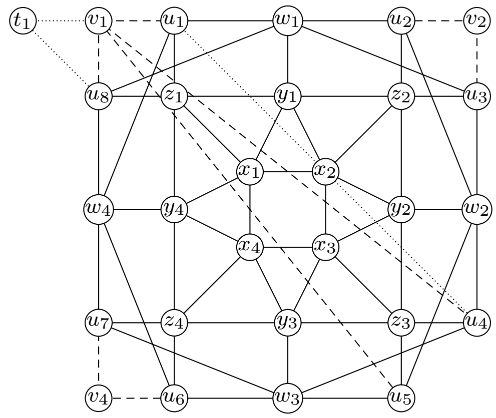


Figure 5.15

By symmetry of the above two cases, we have proved that all v_i are distinct as shown in Fig. 5.16. Consider edge u_1w_1, which is in two C_4 through $w_1u_8v_3u_1$ and $w_1y_1z_1u_1$. Since no C_5 can be formed through u_1w_4 as $u_8 \nsim u_3$ considering edge z_1y_1, edge u_1w_1 must be type-D and forms a C_4 through w_1u_2 or w_1u_3. Since vertices u_2 and u_3 are interchangeable, wlog, the C_4 passes through w_1u_2. Thus u_1, u_2 must have a common neighbor namely t_1, since every existing vertex has degree greater than 3. By rotational symmetry, we obtain $u_3 \sim t_2 \sim u_4$, $u_5 \sim t_3 \sim u_6$, and $u_7 \sim t_4 \sim u_8$. With some quick calculations analogous to the proof that all v_i are distinct, we see that all four t_i are distinct vertices.

Next, we consider edge w_1u_8, which is in two C_4 through u_8z_1 and u_8w_4. Since both w_1 and u_8 are at maximum degree, we must form two C_5 by connecting $t_4 \sim t_2$ and $v_1 \sim t_1$. By rotational symmetry, we have $t_1 \sim t_3, t_1 \sim v_2, t_2 \sim v_2, t_2 \sim v_3, t_3 \sim v_3, t_4 \sim v_1, t_4 \sim v_4$. Now edge $t_2 \sim t_4$ is in four C_5 and must be in two C_4. However, the only way to form a C_4 is by connecting $v_1 \sim v_2$ but then u_1w_1 would be in four C_4, and we have reached a contradiction, concluding the proof.

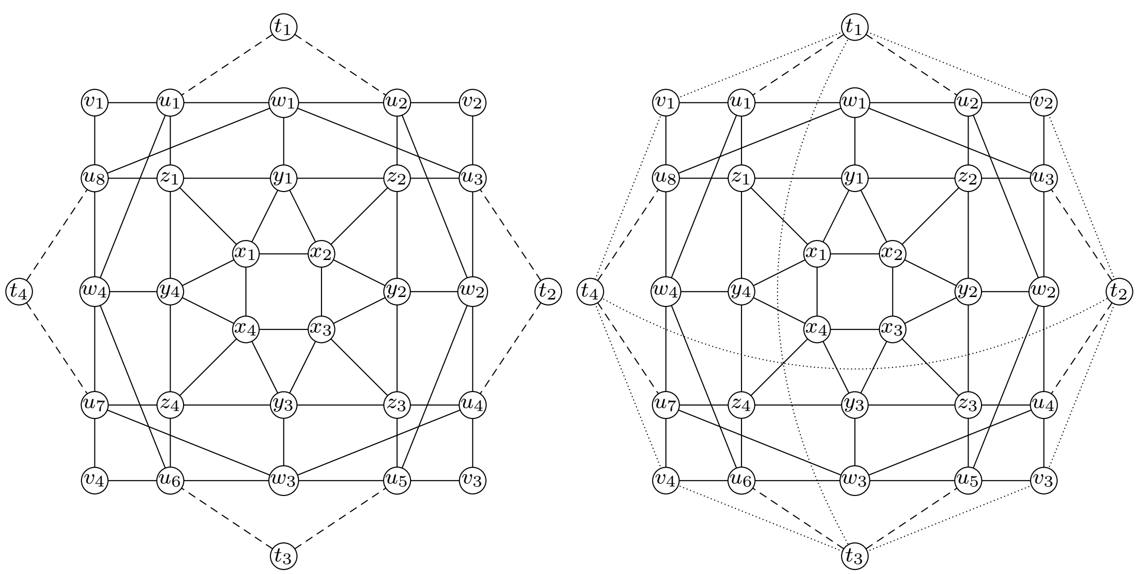


Figure 5.16

□

Lemma 5.4. *If G is a Ricci-flat 5-regular graph, then it does not contain subgraph H shown in Fig. 5.17.*

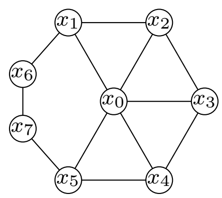


Figure 5.17

Proof. Consider edge x_0x_5 which is already in a C_3 and a C_5. It cannot be type-C because no C_5 can be formed through x_0x_2 without forming a C_4 through x_0x_1. Thus, x_0x_5 must be type-B. Since no C_4 can be formed through x_0x_2 and x_0x_3 given that they must be type-A edges, a C_4 must be formed through x_0x_1. However, if a C_4 is formed through $x_1x_0x_5$, we constructed a forbidden subgraph by Lemma 5.3 and we're done. □

Lemma 5.5. *If G is a Ricci-flat 5-regular graph, then it does not contain subgraph H shown in Fig. 5.18.*

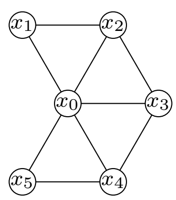


Figure 5.18

Proof. Consider edge x_0x_1. By Lemma 5.1, we have $x_1 \nsim x_3, x_4$. By Lemma 5.2, we have $X_1 \nsim x_5$. Thus, edge x_0x_1 cannot be type-A and must be either type-B or type-C. If x_0x_1. If x_0x_1 is type-C, then it must form a C_5 through x_0x_5, which is impossible by Lemma 5.4. Thus, edge x_0x_1 is type-B and needs a C_4 and C_5. By Lemma 5.3 and Lemma 5.4, no C_4 can be formed through $x0x_5$. However, no C_4 can form through x_0x_3, x_0x_4 given that x_0x_3 and x_0x_4 are type-A edges, and we have reached a contradiction as x_0x_1 cannot be supported on any C_4.

□

Lemma 5.6. *If G is a Ricci-flat 5-regular graph, then it does not contain subgraph H shown in Fig. 5.19.*

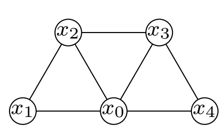


Figure 5.19

Proof. Let the fifth neighbor of x_0 be x_5 shown in Fig.5.20. By Lemma 5.5, $x_5 \nsim x_1, x_4$. Consider edge x_0x_5 which must be type-D or type-E because no C_3 can be formed on x_0x_5. Thus, a C_4 or

C_5 needs to be formed through at least one of x_0x_2 or x_0x_3. Since x_0x_2 and x_0x_3 are type-A edges, forming a C_5 through them implies forming a C_4 through x_0x_1 and x_0x_4. Thus, x_0x_5 must be type-E, and we have $x_1 \sim y_1 \sim x_5$, $x_1 \sim y_2 \sim x_5$, $x_4 \sim y_3 \sim x_5$, and $x_4 \sim y_4 \sim x_5$. Consider edge x_1x_0, which must be type-B since it is in a C_3 and a C_4. The C_5 can either pass through x_0x_4 or x_0x_3.

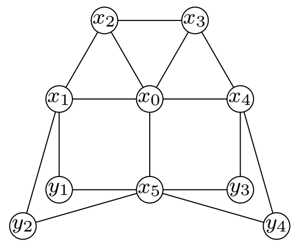


Figure 5.20

1. Suppose x_0x_1 forms a C_5 through x_0x_4, then we must have $x_1 \sim z_1 \sim z_2 \sim x_4$ since x_1 and x_4 cannot connect to any of the existing vertices as in Fig. 5.20. Note that $x_2 \nsim z_1, z_2$ as edge x_2x_0 is type-A, and similarly $x_3 \nsim z_1, z_2$ as edge x_0x_3 is type-A, so x_2, x_3 must have new neighbors. Let $x_2 \sim w_1, w_2$ and $x_3 \sim w_3, w_4$. Consider edge x_3x_4, which is in a C_3 and must be either type-B or type-C as both x_3 and x_4 are at maximum degree. However, x_3x_4 cannot be type-E as no C_5 can be formed through x_3x_2 considering type-A edges x_0x_2 and x_0x_3. Furthermore, no C_4 can be formed through x_3w_3 and x_3w_4 because a C_5 would be formed on the type-A edge x_0x_3. Therefore, x_3x_4 cannot be type-B either, which is a contradiction.

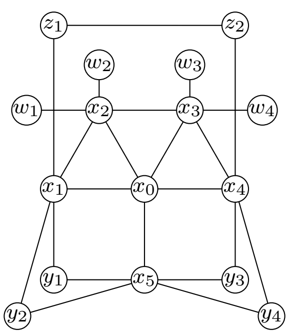


Figure 5.21

2. Thus, x_0x_1 must form a C_5 through x_0x_3. To form the C_5, we must have $x_1 \sim z_1 \sim x_2$. Similarly, x_0x_4 forms a C_5 through x_0x_2, and we have $x_4 \sim z_2 \sim x_3$. Since clearly $x_2 \nsim y_1, y_2, y_3, y_4$, let z_3 be the fifth neighbor of x_2. Note that $\{x_0, x_1, x_2, x_3, z_1\}$ form a subgraph that is isomorphic to the starting configuration. Applying the same argument to x_2z_3, we have $z_3 \sim w_1 \sim z_1$, $z_3 \sim w_2 \sim z_1$, $z_3 \sim w_3 \sim z_2$, and $z_3 \sim z_2$. Now, consider edge x_3w_2, which must be type-E as the subgraph $\{x_2, x_3, z_2, x_4, x_0\}$ is isomorphic to the starting configuration. However, it cannot form any C_3, C_4, C_5 through x_3x_0 as all of its neighbors are at maximum degree, which is a contradiction and we're done.

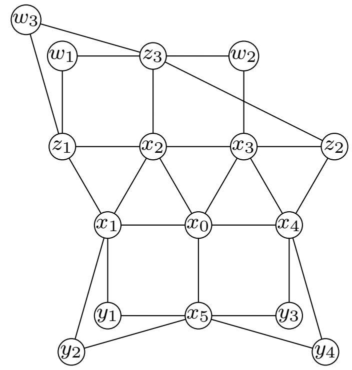


Figure 5.22

□

In the next lemma, we consider graphs with two adjacent C_3.

Lemma 5.7. *If a 5-regular Ricci-flat graph G contains two adjacent C_3, then all other edges of the two C_3 excluding the shared one are type-B and in disjoint C_4.*

Proof. We name the vertices of the subgraph in Fig. 5.23.

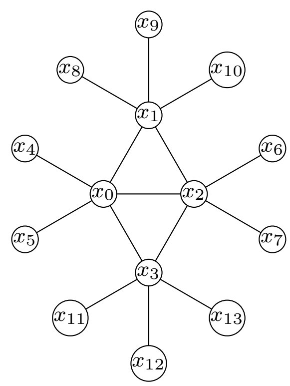


Figure 5.23

By Lemma 5.6, none of edges x_0x_1, x_0x_3, x_2x_1, and x_2x_3 are type-A, thus they are either type-B or type-C. By contradiction, we assume edge x_0x_1 is type-C. Then assume x_0x_4 is in two C_4, then they must pass through edge x_0x_1, x_0x_3 and x_0x_5 which leads to x_0x_1 being in a C_4, and x_0x_1 cannot be type-C, a contradiction. Therefore edge x_0x_4 cannot type-D or type-E, neither is edge x_0x_5. Thus both x_0x_4 and x_0x_5 are in a C_3 which can only be obtained through $x_4 \sim x_5$.

We claim x_0x_4 is not type-C. Otherwise, it must form a C_5 through edge x_0x_2 and then x_2x_3 as $d(x_4, x_6) = d(x_4, x_7) = 3$, this produce a C_4 through edge x_0x_3, a contradiction. Thus x_0x_4 must be type-B. So is x_0x_5.

Since x_0x_1 is not in any C_4, then the C_4 for edge x_0x_4 must pass through edge x_0x_3, then x_0x_3 is type-B that cannot be in two separate C_4. Thus both x_4 and x_5 are adjacent to one neighbor of x_3. Let $x_4 \sim x_{11} \sim x_5$. Then now we have x_4x_5 in two C_3 so this edge is type-A. Let $x_5 \sim x_{12}, x_{13}$ and $x_4 \sim x_{14}, x_{15}$.

Consider edge x_0x_1. Observe that $x_{14}x_4$ or $x_{15}x_4$ cannot form a C_5 through edge x_0x_2, otherwise $x_{14} \sim x_3$ which causes two C_4 on edge x_0x_3. Thus to form a C_5 for edge x_0x_4, it must pass through edge x_0x_1. Wlog, let $x_{14} \sim x_8$. Similarly for edge x_0x_5, let $x_{12} \sim x_9$.

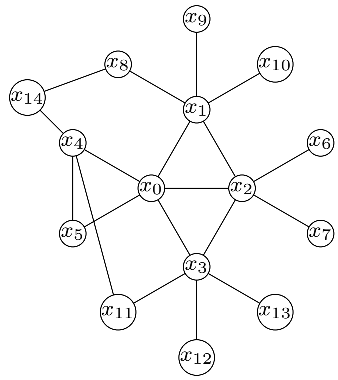


Figure 5.24

Consider the edge x_4x_{11} which is type-B, thus $x_{11} \nsim x_8$. Consider the edge x_5x_{11} which is type-B, thus $x_{11} \nsim x_9$.

We claim that vertex x_{11} does not connect to any of the existing vertices. Assume $x_{11} \sim x_{10}$, observe that vertex x_1 has has distance at most 2 to all neighbors of vertex x_{11}. Thus edge $x_{11}x_{10}$ is neither type-C or type-E. For both cases, it needs a C_4 either from x_4 or x_5. By symmetry of these two vertices, let x_4 be in this C_4, then either $x_{10} \sim x_{14}$ or $x_{10} \sim x_{15}$. However both edges x_4x_{11}, x_5x_{11} are type-B that does not needs two C_4. A contradiction. Since $d(x_5, x_6) = d(x_5, x_7) = 3$, then $x_{11} \nsim x_6$ and $x_{11} \nsim x_7$. Thus vertex x_{11} has two new vertices as its neighbors. Let $x_{11} \sim x_{16}$ and $x_{11} \sim x_{17}$.

Observe edge x_4x_{14}. Clearly it is not type-A . Since it cannot be in any C_4 through either x_4x_0, x_4x_5 or x_4x_{11}, then it is not type-B, type-D or type-E. It is not type-C, otherwise, the C_5 passes through edge x_4x_5 must pass through either x_{11} or x_{16}, however both cases cause two separate C_4 on edge x_4x_{11}. A contradiction. Thus all edges $x_0x_1, x_0x_3, x_2x_1, x_2x_3$ are type-B.

As edge x_0x_1 in a C_4. we claim it is not formed by $x_3 \sim x_8$. Otherwise, none of vertices $\{x_4, x_5, x_6, x_7\}$ can be adjacent to x_9 or x_{10} or any new neighbor of vertex x_3. We claim $x_4 \sim x_8$. Otherwise , there is no C_5 supported on edge x_0x_4 passing through the edge x_0x_2, together with no C_4 supported on edge x_0x_4 passing through the edge x_0x_1 or x_0x_3, then edge x_0x_4 is not any good type. Similarly, we have all x_5, x_6, x_7 are adjacent to x_8, which causes the degree $d_8 = 6$. A contradiction.

Therefore, we must have the subgraph shown in Fig. 5.25.

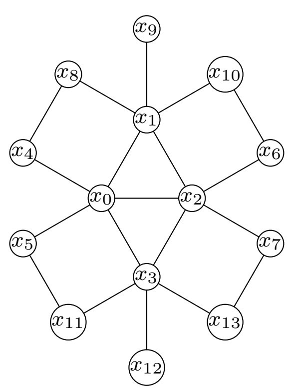


Figure 5.25

□

Lemma 5.8. *If a Ricci-flat 5-regular graph G contains a pair of adjacent C_3, then both vertices of the shared edge are not in any other C_3.*

Proof. We use the same numbering of vertices as above, then the lemma essentially states that edges x_0x_4, x_0x_5, x_2x_6, x_2x_7 are not in C_3.

By Lemma 5.7, we have $x_3 \nsim x_8, x_9, x_{10}$. Let $x_3 \sim x_{11}, x_{12}, x_{13}$. For edges x_0x_1, x_2x_1 in C_4, Wlog, let $x_4 \sim x_8, x_6 \sim x_{10}$. In the following, we assume vertex $x_4 \nsim x_9, x_{10}$. Note if we cannot construct a Ricci-flat graph under this assumption, then vertex x_4 must be adjacent to at least two of $\{x_8, x_9, x_{10}\}$, and by symmetry of x_4 and x_6, x_6 is also adjacent to at least two of $\{x_8, x_9, x_{10}\}$ which causes $d(x_4, x_6) = 2$, a contradiction.

For a contradiction of our result, we assume edge x_0x_4 is in a C_3, then it has to be $x_5 \sim x_4$ and x_0x_4 is type-B. Note the edge x_0x_3 needs a C_4 either from x_0x_4 or x_0x_5, since there cannot be two separate C_4 supported on x_0x_4, then it must be through x_0x_5. Wlog, let $x_5 \sim x_{11}$. Clearly, x_0x_5 is also type-B. It is easy to see that $x_5 \nsim x_8, x_9, x_{10}$. As $x_4 \nsim x_9, x_{10}, x_{11}, x_{12}, x_{13}$, then the C_5 supported on x_0x_4 cannot pass through x_0x_2 and must pass through edge x_0x_3 directly. We assume the C_5 passes through $x_4x_0x_3x_{11}$. Let $x_4 \sim x_{14} \sim x_{11}$, let $x_4 \sim x_{15}$.

1. $x_5 \sim x_{12}$. Note $x_5 \nsim x_{14}$, otherwise a subgraph in Lemma 4.6 is generated. Let $x_5 \sim x_{16}$. Then $d(x_2, x_{15}) = 3$ for edge x_0x_4, $d(x_1, x_{16}) = 3$ for edge x_0x_5, $d(x_1, x_{13}) = 3$ for edge x_0x_3. Note $x_8 \nsim x_{11}, x_{12}, x_{13}, x_{16}$ considering edge x_4x_5. Now the edge x_0x_1 needs a C_5.

   - If $x_9 \sim x_{12}$, we have $d(x_1, x_{16}) = d(x_2, x_{16}) = 3$ for edge x_0x_5, $d(x_5, x_{10}) = 3$ for edge x_0x_1, $d(x_4, x_{13}) = 3$ for edge x_0x_3 and $d(x_2, x_{16}) = 3$ for edge x_0x_5. If $x_{11} \sim x_{12}$, let $x_{12} \sim x_{17}$, then $d(x_{17}, x_{13}) = 3$ for edge x_3x_{12}. Consider edge x_3x_{13}, the largest cycle passing through it are C_4, C_5, C_5, C_5 each with $x_3x_2, x_3, x_{12}, x_3x_0, x_3x_{11}$. Thus edge x_3x_{13} is not any good type. A contradiction. Similarly, $x_{12} \nsim x_{13}$ as largest cycle passing through x_3x_{11} are also C_4, C_5, C_5, C_5. If $x_{11} \sim x_{13}$, then largest cycle passing through x_3x_{12} are also C_4, C_5, C_5, C_5. Therefore, none of edges $\{x_3x_{11}, x_3x_{12}, x_3x_{13}\}$ is in C_3.

     Consider edge $x_{13}x_3$, if it is type-D, then there are two C_4 passing through x_3x_{11}, x_3x_{12}. Let $x_{13} \sim x_{17} \sim x_{12}$ and $x_{13} \sim x_{18} \sim x_{111}$, then the edge x_3x_{12} is type-E as it is in two C_4 and one C_5, and it need a C_5 passing through edge x_3x_{11}, let $x_{12} \sim x_{19} \sim x_{18}$. Observe that edge x_3x_{11} is also type-E that must have $x_9 \sim x_{11}$ for the second C_5. Consider edge x_5x_{16}, $x_{16} \sim x_{12}$ as x_{12} achieve the maximal degree, $x_{16} \sim x_{13}$ considering x_5x_{11} or x_3x_{13}. Thus there cannot be any C_5 through $x_{15}x_5x_0$ which implies $x_{16}x_5$ is type-D that needs three C_4. To avoid two C_4 on x_4x_5, it needs $x_{16} \sim x_{14}$. However, there cannot be any C_4 through edge x_5x_{11}. A contradiction. Thus edge $x_{13}x_3$ is type-E which needs a C_5 through x_3x_0, this can only be formed by $x_{13} \sim x_{16}$.

     If further assume $x_{16} \sim x_{12}$. The edge x_5x_{12} is in $C_3 = x_{16}x_5x_{12}x_{16}$, thus either type-B or type-C. Then x_{12} is not adjacent to any existing vertices for the edge x_0x_1 and x_5x_{12}. Let $x_{12} \sim x_{17}$. Then consider edge x_3x_{12}, if (a) $x_{12}x_{16}$ contributes in the $C_4 : x_3x_{12}x_{16}x_{13}x_3$, then together with the other $C_4 : x_3x_{12}x_0x_5x_3$, it is type-E that need one more C_5 through x_3x_{11} and $x_{12}x_7$, that is $d(x_{11}, x_{17}) = 2$. By edge x_3x_2, we must have wlog $x_{13} \sim x_7$, note then there is no C_5 supported on $x_{11}x_3$ passing through x_3x_2, then $x_{11}x_3$ must be type-B that can only be formed by $x_{11} \sim x_{13}$. However, the edge x_3x_{13} would be in one C_3 and two C_4, a contradiction. Thus (b) for edge x_3x_{12}, $x_{12}x_{16}$ contributes in the $C_5 : x_3x_{12}x_{16}x_5x_{11}x_3$, then it needs $x_{13} \sim x_{17}$. Observe edge x_3x_{13} which is in two C_4 through x_3x_2, x_3x_{12} and one C_5 through x_3x_0, thus it is type-E that passes through a C_5 through x_3x_{11}. Since x_3x_{11} is in the $C_4 : x_3x_{11}x_5x_{12}x_3$ and $C_5 : x_3x_{11}x_{14}x_4x_0x_3$ and there is no new C_5 through $x_{11}x_3x_2$, then x_3x_{11} must be type-B that form a C_3 through x_3x_{12}, however, this cannot be true as vertex x_{12} arrives the maximal degree. Then $x_{16} \nsim x_{12}$, then $x_{16}x_5$ cannot be in any C_3 thus is type-E that forms C_5 through x_5x_4 and forms two C_4 through x_5x_{12} an d x_5x_{11}. Note if $x_{13} \sim x_{17} \sim x_{12}$, then $d(x_{12}, x_{11}) = 3$ for edge x_3x_{12}. While under this situation, edge x_3x_{11} cannot be any good type. Thus for edge x_3x_{13}, we must have $x_{11} \sim x_{18} \sim x_{13}$ and $x_{12} \sim x_{17} \sim x_{19} \sim x_{13}$. Then for edge x_3x_{11}, the fifth neighbor of vertex x_{11} must have distance 2 to vertex x_2 which can be achieved by $x_{11} \sim x_{20} \sim x_7$. Now consider edge x_3x_{12}, it must be type-E that needs $x_{12} \sim x_{18}$. Then we consider the edge x_5x_{16}, which has to be type-E that need two C_4 through x_3x_{11} and x_3x_{12}, the only way to attain this is $x_{16} \sim x_{14}$, $x_{16} \sim x_{17}$. Let $x_{16} \sim x_{21} \sim x_{15}$. NOw we consider edge $x_{11}x_{20}$, as the neighbor of x_{11}: x_5, x_0 have maximal degree and their neighbors also have maximal degree, there is no way to construct a good type for $x_{11}x_{20}$. Thus we conclude $x_9 \sim x_{12}$.

- Then the C_5 passing through x_0x_1 using edge x_0x_5. Let $x_5 \sim x_{16} \sim x_9$. However, there would be two C_5 supported on edge x_0x_5, a contradiction.

2. Thus $x_5 \nsim x_{12}, x_{13}$. Then the C_5 supported on x_0x_5 must pass through edge x_0x_1 directly. Let $x_5 \sim x_{16} \sim x_9$. Now we need a C_5 supported on x_0x_3, still we need to consider if $x_9 \sim x_{12}$ works. However, we found that a Ricci-flat graph based on this subgraph cannot be constructed with the aid of calculator [5].

Thus the C_5 for edge x_0x_4 passes through $x_4x_0x_3x_{12}$. Let $x_4 \sim x_{14} \sim x_{12}$, let $x_4 \sim x_{15}$. By symmetry, let $x_5 \sim x_{16} \sim x_9$ and $x_5 \sim a$. Note if $a = x_{15}$, then edge x_4x_5 is also type-A edge which should have a same local structure as x_0x_2, and a Ricci-flat graph cannot be obtained ☐

Lemma 5.9. *If G is a Ricci-flat 5-regular graph, then it contains an edge that is not in any triangle, i.e., a type-D or type-E edge.*

Proof. If G contains type-A edges, the conclusion is obvious by Lemma 5.8; if G does not contain type-A edge, the same argument in Lemma 4.3 applies, and G cannot consist of only type-B and type-C edges. ☐

6 Future directions

Through extensive search and construction, we have not yet found another Ricci-flat 5-regular graph that is not isomorphic to RF^5_{72} or a Cartesian product of Ricci-flat cubic graphs and cycles. Therefore, our main conjecture is the following:

Conjecture 1. *If G is a Ricci-flat 5-regular graph, then G is either isomorphic to RF^5_{72} or a Cartesian product of the Petersen graph, the Triplex graph, or the dodecahedral graph with a cycle of length at least 6, or the infinite path.*

To break up the main conjecture into smaller manageable pieces, we have the following conjectures.

Conjecture 2. *If G is a Ricci-flat 5-regular graph, then G does not contain adjacent triangles.*

Conjecture 3. *If G is a Ricci-flat 5-regular graph, then G does not contain vertex-sharing triangles.*

Conjecture 4. *If G is a Ricci-flat 5-regular graph, then G does not contain vertex-disjoint triangles.*

As a result of combining Conjectures 2–4, we have:

Conjecture 5. *If G is a Ricci-flat 5-regular graph, then G has girth 4, i.e. it does not contain triangles.*

For Ricci-flat 5-regular graph of girth 4, it will be important to prove or disprove the following conjectures:

Conjecture 6. *If G is a Ricci-flat 5-regular graph, then G contains type-E edges.*

Conjecture 7. *If G is a Ricci-flat 5-regular graph that contains only type-E edges, then G is isomorphic to RF^5_{72}.*

Conjecture 8. *If G is a Ricci-flat 5-regular graph and contains both type-D and type-E edges, then G is either isomorphic to RF^5_{72} or a Cartesian product of the Petersen graph, the Triplex graph, or the dodecahedral graph with a cycle of length at least 6, or the infinite path.*

Conjectures 6-8, if proven to be true, will finish the proof of our main conjecture.

We also have the following over-arching questions concerning Ricci-flat regular graphs of arbitrary degree and their automorphism groups.

Question 1. *Does there exist a Ricci-flat arc-transitive graph of every degree?*

Question 2. *What about edge-transitive, vertex-transitive, and symmetric graphs?*

Question 3. *What can be said in general about the automorphism group of a Ricci-flat regular graph?*

Acknowledgements

The author would like to thank Ms. Shuliang Bai for proposing the topic of research and advising the project. The author would also like to thank family, friends, and teachers for their support.

References

[1] Shuliang Bai, Linyuan Lu, and Shing-Tung Yau. *Ricci-flat graphs with maximum degree at most 4*.

[2] David Bourne et al. “Ollivier–Ricci Idleness Functions of Graphs”. In: *SIAM Journal on Discrete Mathematics* 32 (Apr. 2017). DOI: 10.1137/17M1134469.

[3] David Cushing, Shiping Liu, and Norbert Peyerimhoff. “Bakry-Émery curvature functions of graphs”. In: *Canadian Journal of Mathematics* (June 2016). DOI: 10.4153/CJM-2018-015-4.

[4] David Cushing et al. *Ricci-flat cubic graphs with girth five*. Feb. 2018. arXiv: 1802.02982 [math.CO].

[5] David Cushing et al. “The Graph Curvature Calculator and the curvatures of cubic graphs”. In: *Experimental Mathematics* (Dec. 2017). DOI: 10.1080/10586458.2019.1660740.

[6] Weihua He et al. *Ricci-flat graphs with girth four*. July 2018. arXiv: 1807.07253 [math.CO].

[7] Yong Lin, Linyuan Lu, and S T. Yau. “Ricci-flat graphs with girth at least five”. In: *Communications in Analysis and Geometry* 22 (Jan. 2013). DOI: 10.4310/CAG.2014.v22.n4.a3.

[8] Yong Lin, Linyuan Lu, and Shing-Tung Yau. “Ricci curvature of graphs”. In: *Tohoku Mathematical Journal - TOHOKU MATH J* 63 (Dec. 2011). DOI: 10.2748/tmj/1325886283.

[9] Shiping Liu, Florentin Münch, and Norbert Peyerimhoff. “Bakry-Emery curvature and diameter bounds on graphs”. In: *Calculus of Variations and Partial Differential Equations* 57 (Apr. 2018). DOI: 10.1007/s00526-018-1334-x.

[10] Yann Ollivier. “Ricci curvature of Markov chains on metric spaces”. In: *Journal of Functional Analysis* 256 (Feb. 2009), pp. 810–864. DOI: 10.1016/j.jfa.2008.11.001.

[11] Yann Ollivier. “Ricci curvature of metric spaces”. In: *Comptes Rendus Mathematique* 345.11 (2007), pp. 643–646. ISSN: 1631-073X. DOI: https://doi.org/10.1016/j.crma.2007.10.041. URL: http://www.sciencedirect.com/science/article/pii/S1631073X07004414.

A Adjacency list for RF_{72}^{5}

```
1:[2,3,4,5,8],
2:[1,12,6,7,9],
3:[11,1,13,7,18],
4:[1,14,6,19,10],
5:[11,1,16,28,20],
6:[2,4,17,29,21],
7:[22,2,3,15,26],
8:[1,23,16,30,10],
9:[2,25,17,30,41],
10:[4,27,8,31,42],
11:[24,3,5,38,32],
12:[33,2,25,15,20],
13:[3,36,26,19,53],
14:[4,37,27,28,54],
15:[55,12,38,39,7],
16:[34,5,40,8,43],
17:[44,35,6,9,31],
18:[23,45,24,3,36],
19:[13,46,4,37,21],
20:[12,5,38,49,62],
21:[26,6,50,19,64],
22:[45,48,39,7,41],
23:[18,40,51,8,65],
24:[11,44,35,18,40],
25:[12,56,47,52,9],
26:[13,57,48,7,21],
27:[55,34,14,39,10],
28:[34,14,58,5,49],
29:[33,35,6,50,54],
30:[59,51,8,9,31],
31:[66,17,61,30,10],
32:[11,35,58,61,53],
33:[12,69,29,62,52],
34:[57,48,16,27,28],
35:[24,17,29,52,32],
36:[56,13,18,52,63],
37:[56,14,47,60,19],
38:[11,66,15,61,20],
39:[22,47,15,27,42],
40:[23,24,16,50,64],
41:[22,67,47,59,9],
42:[46,39,61,10,65],
43:[48,59,16,50,62],
44:[66,67,24,17,64],
45:[22,67,18,63,65],
46:[47,72,19,42,53],
47:[46,25,37,39,41],
48:[22,34,26,63,43],
49:[56,68,28,63,20],
```

```
50:[60,29,40,21,43],
51:[23,56,68,60,30],
52:[33,35,25,36,53],
53:[13,46,70,52,32],
54:[14,58,69,60,29],
55:[66,57,69,15,27],
56:[25,36,37,49,51],
57:[55,34,26,70,64],
58:[68,70,28,32,54],
59:[60,71,30,41,43],
60:[37,59,50,51,54],
61:[68,38,31,42,32],
62:[33,71,63,20,43],
63:[45,36,48,49,62],
64:[44,57,72,40,21],
65:[23,45,68,72,42],
66:[44,55,38,71,31],
67:[44,45,71,72,41],
68:[58,49,61,51,65],
69:[33,55,70,71,54],
70:[57,58,69,72,53],
71:[66,67,69,59,62],
72:[67,46,70,64,65]
```