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Research Report

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Title of Research Report

Construction of Higher Universal Covering Spaces

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Construction of Higher Universal Covering Spaces

Alexander Goo

Abstract

The universal covering space is a cover that is simply connected. A well known result states that for any connected, locally path-connected and semi-locally simply connected, the universal cover exists and is unique. One key property of the universal cover is that its homotopy groups are the same as the space that we started off with with the key exception of the fundamental group which vanishes. In this project, we aim to generalize this notion of a universal covering to higher homotopy groups, and find an appropriate notion of covering spaces to kill off higher homotopy groups.

Keywords: Algebraic Topology, Homotopy groups, Category theory, Covering spaces

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1. Mr Chai Ming Huang Royce, NUS High School of Mathematics and Science for providing guidance and direction.

Commitments on Academic Honesty and Integrity

We hereby declare that we

1. are fully committed to the principle of honesty, integrity and fair play throughout the competition.
2. actually perform the research work ourselves and thus truly understand the content of the work.
3. observe the common standard of academic integrity adopted by most journals and degree theses.
4. have declared all the assistance and contribution we have received from any personnel, agency, institution, etc. for the research work.
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1 Introduction

Galois correspondence has been around since Galois which in a vague sense, relates 'extensions' of an object to the 'subobjects' of another object. The typical first example seen is the correspondence between field extensions and their Galois groups, subgroups of the absolute Galois group. In topology, a similar phenomenon occurs with covering spaces and their group of deck transforms, given by subgroups of π_1. While this idea has been known even before topology was formalized by mathematicians like Klein and Riemann[1], not much is known about if we can generalize this idea to the higher homotopy groups π_n, in particular, due to the intricate yoga required to do such a generalization as the definitions of n-categories are still not very well understood and agreed on. In this paper, we propose an approach to generalize covering spaces in a way that is natural in a concrete sense.
One may quickly recognise that the Postnikov and Whitehead towers in homotopy theories have a very similar idea. More specifically, they are:

Definition 1.0.1. *[2] A Postnikov tower of a path connected space X is an inverse system of truncations*

$$\cdots \to X_n \to \cdots \to X_2 \to X_1 \to X_0 = \{*\}$$

such that the limit is X, $\pi_{>n}(X_n) = 0$ and the natural map $X \to X_n$ induced by the limit gives us an isomorphism of homotopy groups $\pi_{\leq n}(X) \to \pi_{\leq n}(X_n)$. Furthermore, each map in the system is a fibration with the homotopy fibre of $X_n \to X_{n-1}$ being $K(\pi_n(X), n)$, the Eilenberg-MacLane space.

This is most naturally constructed by considering the homotopy hypothesis and taking the truncations of the fundamental ∞-groupoid. However, covering spaces are in some sense 'opposite' to this chain, hence we can dualize it to obtain the Whitehead tower:

Definition 1.0.2. *A Whitehead tower of a path connected space X is (trivial directed system of) maps*

$$\cdots \to X_n \to \cdots \to X_2 \to X_1 \to X_0 = X$$

such that $\pi_{\leq n}(X_n) = 0$ and the natural map $X_n \to X$ induced by the diagram above gives us an isomorphism of homotopy groups $\pi_{>n}(X) \to \pi_{>n}(X_n)$. Furthermore, each map in the system is a fibration with the homotopy fibre of $X_n \to X_{n-1}$ being $K(\pi_n(X), n-1)$, the Eilenberg-MacLane space.

Such a system can be constructed by noticing that taking the homotopy fibre of $X \to X_n$ in the Postnikov system flips which homotopy groups are killed. While X_1 is indeed the universal cover, such constructions are only up to homotopy and does not give us the Galois correspondence and the concrete constructions in classical algebraic topology. However, it does suggest to us to study the most natural way to construct these towers, using the fundamental ∞-groupoid and taking truncations.
We note that throughout this paper, we are not concerned with set-theoretic size issues. We assume the reader is familiar with basic definitions in topology and category theory and we shall only review definitions that aren't covered in introductory courses in topology and category theory, such as from Munkres' Topology [3], Riehl's Category Theory in Context [4] and Kashiwara and Schapira's Categories and Sheaves [5].

1.1 Review on topology and homotopy theory

One can find a more detailed exposition to homotopy theory in tom Dieck's algebraic topology book [6] and Jeffrey Storms' book on classical homotopy theory [7].
We start off by giving some conventions that we will use. We use the following notations for standard subspaces of $\mathbb{R}^n$:

Notation	Name
$\mathbb{R}^n$	Euclidean space
$D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$	n-disk
$S^n = \partial D^{n+1} = \left\{x \in \mathbb{D}^{n+1} : \|x\| = 1\right\}$	n-sphere
$E^n = D^n - S^{n-1}$	n-cell
$I^n = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$	n-cube
$\partial I^n = \{x \in I^n : \exists i, x_i = 0, 1\}$	boundary of I^n
$\lvert\Delta^n\rvert = \left\{x \in \mathbb{R}^{n+1} : x_i \geq 0, \sum_i x_i = 1\right\}$	n-simplex
$\lvert\partial\Delta^n\rvert = \{x \in \Delta^n : \exists i, x_i = 0\}$	Boundary of n-simplex

Suppose $p : E \to B$ is surjective with $p^{-1}(b) \cong F$ for all $b \in B$ and $U \subset B$ is open, we use the following conventions:

- **Trivialization** of p over U is a homeomorphism $p^{-1}(U) \to U \times F$
- p is **locally trivial** if a open covering $\mathcal{U}$ exists where a trivialization of p over $U \in \mathcal{U}$ exists for all U
- $\mathcal{U}$ is a **bundle chart**
- F is the **typical fibre**
- p is **trivial over** U if a bundle chart over U exists
- **Bundles/Fibre bundles** are locally trivial maps

Covering space/Covering of B is a locally trivial trivial map $p : E \to B$ with discrete fibres

- If $\phi_U : p^{-1}(U) \to U \times F$ is a trivialization, then $\phi_U^{-1}(U \times \{*\})$ are the **sheets** over U
- If $|F| = n$, then p is a n-fold covering
- A **trivial covering** is the covering $p : B \times F \to B$
- U is **admissible** or **evenly covered** if a trivialization exists
- E is the **total space** and B is the **base space**

Definition 1.1.1. *Exponential objects or Internal homs of a category $\mathcal{C}$, denoted by Z^Y or $\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z)$, are objects that satisfy the following isomorphism:*

$$\mathrm{Hom}_{\mathcal{C}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}\left(X, Z^Y\right) = \mathrm{Hom}_{\mathcal{C}}\left(X, \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z)\right)$$

The category Top does contain exponential objects, in particular

Theorem 1.1.2. *If X is locally compact, then $\underline{\mathrm{Hom}}_{\mathrm{Top}}(X, -)$ exists and is given by the compact open topology on $\mathrm{Hom}_{\mathrm{Top}}(X, -)$*

We occasionally denote $\underline{\mathrm{Hom}}_{\mathrm{Top}}(X, Y)$ by Y^X if there will be no confusion.
We now recall the basic definitions of a homotopy

Idea. Homotopy intuitively encodes the notion of 'moving from one point/function to another smoothly' in a formal fashion.

Definition 1.1.3. *Given morphisms $f, g : X \to Y$ of topological spaces, a homotopy is a morphism $H : X \times I \to Y$ such that $H(-, 0) = f(-), H(-, 1) = g(-)$.*

Definition 1.1.4. *The category* hTop *is constructed by taking the category* Top *and identifying homotopic morphisms as the same.*

Definition 1.1.5. *S^n are naturally group objects, hence we define the groups $\pi_n(X) := \mathrm{Hom}_{\mathrm{hTop}}(S^n, X)$. For $n = 0$, $\pi_0(X)$ is given by quotienting the points of the space X by homotopy, i.e. it counts the number of path connected components.*

Definition 1.1.6. *A homotopy between points $x, y \in X$ is a morphism $f : I \to X$ such that $f(0) = x, f(1) = y$.*

An important tool that we will use is the theory of fibrations, in particular,

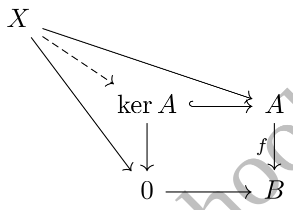
Idea. In abelian categories, such as the category of

- Abelian groups
- Commutative rings
- Left/Right modules over a ring

the idea of a kernel and a cokernel is well defined and unique. Categorically, we define it

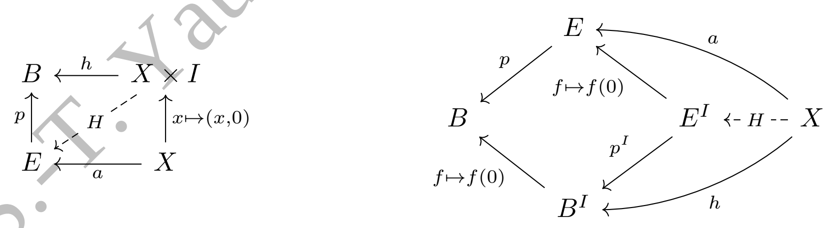
$$\ker f = \lim \left(A \xrightarrow{f} B \leftarrow 0 \right)$$

Concretely, this means that for any object X with morphisms that makes the following diagram commute:



A unique morphism given by the dotted arrow exists such that the whole diagram commutes. Unfortunately Top isn't such a nice category, and we can't enforce the uniqueness conditions, both for the kernel and for the morphism. Hence we weaken the notion to get a fibration where the kernel, in this case the fibre, exists up to homotopy and the induced morphism is no longer unique.

Definition 1.1.7. *$p : E \to B$ has the **homotopy lifting property** (HLP) for X if for every h, a, there exists some H such that the (equivalent) diagrams commutes:*



*H is an **lifting** of h with **initial conditions** a.*

Definition 1.1.8. *If p has HLP for every space X, then it is a **fibration**.*

Intuitively, fibrations are somewhat like epimorphisms with kernels. An important class of fibrations are those coming from fibre bundles:

Lemma 1.1.9. *If $p : E \to B$ is a fibre bundle, then it is a fibration 'with kernel $p^{-1}(b)$'. More precisely, the sequence $p^{-1}(b) \to E \to B$ is exact.*

and we get the exact sequence

Theorem 1.1.10 (Puppe sequene)**.** *If $F \to E \to B$ is a fibre sequence, then the sequence*

$$\cdots \to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

is exact.

and this gives us the exact sequence

Corollary 1.1.11.

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \pi_{n-1}(E) \to \pi_{n-1}(B) \to \dots$$

1.2 Review on simplicial sets

We often use simplicial sets in homotopy, and in a strict sense, this allows us to model every topological space up to weak equivalence. We give the definitions of these sets here. One can find more information in Goerss and Jardine's book on Simplicial Homotopy theory [8] as well as the lecture notes by Prof. Dr. Tobias Dyckerhoff from Universität Hamburg [9].

Idea. In constructing (co)homology groups, we often use complexes to turn a topological space into a hopefully finite combinatorial object that is easy to work with, making (co)homology an extremely powerful computational tool. One way to do this is using simplicial complexes, where we assign (co)homology groups based on a chosen triangulation of a topological space. However, choosing an arbitrary fixed structure often leads to issues for less well-behaved spaces, hence one generalizes this idea into simplicial sets, where we aim to provide a model of the category Top via an 'abstract triangulation'.

Definition 1.2.1. *Every ordinal α can naturally be realized as a poset category. This leads us to define the category $[\alpha]$ where objects are the elements of α and morphisms are natural transformations of these poset categories, equivalently, they are weakly monotone maps.*

Definition 1.2.2. *The category Δ is defined as the ordinal category $[\omega]$.*

Definition 1.2.3. *The category of simplicial sets* sSet *is defined as* $\mathrm{Psh}(\Delta) = \underline{\mathrm{Hom}}_{\mathrm{Cat}}(\Delta^{\mathrm{op}}, \mathrm{Set})$.

Definition 1.2.4. *For $S \in P([n+1])$, define the simplicial set Δ^S as*

$$\Delta^S([m]) = \{f \in \mathrm{Hom}_\Delta([m],[n]) \,|\, \exists s \in S \;\; f([m]) \subset s\}$$

Definition 1.2.5. *From the previous definition, we obtain the following common simplicial sets:*

- *Standard n-simplex:* $\Delta^{P([n+1])} = \mathrm{Hom}_\Delta(-,[n]) := \Delta^n$
- *Boundary of n-simplex:* $\Delta^{P([n+1])-\{[n+1]\}} := \partial\Delta^n$
- *ith horn of n-simplex:* $\Delta^{P([n+1])-\{[n+1],[n+1]-\{i\}\}} := \Lambda^n_i$

Definition 1.2.6. *Define the coface map $\delta^n_k : [n-1] \to [n]$ as the unique injective map in Δ that misses k and the codegeneracy map $\sigma^n_k : [n+1] \to [n]$ as the unique surjective map in Δ that hits k twice. Under a simplicial set S, these gets turned into the face map $d^n_k : S([n]) \to S([n-1])$ and the degeneracy map $s^n_k : S([n]) \to S([n+1])$.*

Lemma 1.2.7. *It turns out that all morphisms in Δ are generated by all the δ^n_k and σ^n_k, so by defining the face and degeneracy maps of a simplicial set, it is the same as defining the whole simplicial set. This makes defining simplicial sets explicitly easier.*

Definition 1.2.8. *For any category $\mathcal{C}$, define the nerve to be the simplicial set $N(\mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}}([n],\mathcal{C})$ and the image of morphisms are given by composition.*

Lemma 1.2.9. *For a simplicial set, we have*

$$K \cong \mathrm{colim}_{\Delta^n \to K} \Delta^n$$

Definition 1.2.10. *Given a simplicial set K, define the geometric realization to be*

$$|K| \cong \mathrm{colim}_{\Delta^n \to K} |\Delta^n|$$

where $|\Delta^n| = \left\{x \in \mathbb{R}^{n+1} | x_i \geq 0, \sum_i x_i = 1\right\}$ *is the standard n-simplex.*

1.3 Review on infinity categories

We will use infinity category to mean $(\infty, 1)$-categories in this paper. We suggest checking out Moritz Groth's introduction on arxiv [10] and André Joyal's book [11] before looking at Lurie's books [12, 13].

Idea. Classically, the fundamental group and the higher homotopy groups require the consideration of a base point and have a rather awkward construction and lose out on information about the other homotopy groups or other possible base points. This can be resolved by considering the fundamental infinity groupoid of a space. This 'infinity category' has objects as the points of a topological space, morphisms as homotopies between the points, morphisms between morphisms as the homotopies between homotopies and so on. It turns out that in a strict sense, this encodes all the information we need about the topological space, known as the 'homotopy hypothesis'[14]. We note that this 'hypothesis' is more of a guiding principle as to how we should define various notions regarding infinity categories.

Definition 1.3.1. *A simplicial set $\mathcal{C}$ is an $(\infty, 1)$-category if for all $0 < i < n$, every inner horn $\Lambda^n_i \to \mathcal{C}$ can be extended to a n-simplex $\Delta^n \to \mathcal{C}$, meaning the dotted morphism exists such that the diagram below commutes (but may not be unique):*

$$\begin{array}{ccc} \Lambda^n_i & & \\ \big\downarrow & \searrow & \\ \Lambda^n & \dashrightarrow & \mathcal{C} \end{array}$$

Definition 1.3.2. *If every horn (including $i = 0, n$) can be extended, the simplicial set is called a Kan complex and is also a ∞-groupoid. While it is indeed an $(\infty, 1)$-category, it is also a $(\infty, 0)$-category as all 1-morphisms are invertible up to homotopy.*

Example. One can easily verify that $N(\mathcal{C})$ is an $(\infty, 1)$-category for any category $\mathcal{C}$. Furthermore, the extension in the definition above is unique by composition.

Example. Consider the functor

$$\Pi(X)([n]) = \mathrm{Hom}_{\mathrm{Top}}\left(|\Delta^n|, X\right)$$

Every horn has a filling (even if $i = 0, n$) but this filling may not be unique. This functor is an $(\infty, 0)$-category and is known as the fundamental groupoid of X.

Definition 1.3.3. *For a simplicial set K and a $(\infty, 1)$-category $\mathcal{C}$, we define the internal hom of* sSet *to be*

$$\underline{\mathrm{Hom}}_{\mathrm{sSet}}\left(K, \mathcal{C}\right)([n]) = \mathrm{Hom}_{\mathrm{sSet}}\left(\Delta^n \times K, \mathcal{C}\right)$$

This allows us to define functors and natural transformations of $(\infty, 1)$ categories as the images of $[0]$ and $[1]$ of this simplicial set respectively.

Theorem 1.3.4. *The simplicial set defined in the previous definition is an $(\infty, 1)$-category.*

Definition 1.3.5. *A n-morphism from x to y in an $(\infty, 1)$-category $\mathcal{C}$ is a map $\Delta^{n+1} \to \mathcal{C}$ of simplicial sets such that Δ^{n+1} sends $\Delta^{P([n+1])}$ to x and $\Delta^{\{n+1\}}$ to y.*

Remark 1.3.6. In this way, we have constructed a category where all n-morphisms for $n > 1$ are invertible 'up to homotopy', thus the name $(\infty, 1)$ category.

Definition 1.3.7. *The n-truncation of an infinity category $\mathcal{C}$, denoted by $\tau_{\leq n}\mathcal{C}$, is constructed by identifying n morphisms that are connected by an invertible $n+1$ morphism. We note that this is still a proper $(\infty, 1)$-category, but all the higher simplices are degenerate. Such a category is known as a $(n, 1)$-category and the category of all $(n, 1)$-categories is given by n-*Cat.

Example. We have $\tau_{\leq n}\Pi(X) = \Pi_n(X)$, the fundamental n-groupoid of a space. In particular, $\Pi_0(X)$ is a discrete category where points are path connected components and $\mathrm{Hom}_{\Pi_1(X)}(x, x) = \pi_1(X, x)$. This also allows us to define $\mathrm{Hom}_{\Pi_n(X)}(x, x) = \pi_n(X, x)$.

Example. Recall that we have a very natural motivation to give Top and $(\infty,1)$-category structure: 1-morphisms are continuous functions, 2-morphisms are homotopies, 3-morphisms are homotopies between homotopies, etc. To construct Top as a simplicial set, we define n-simplices to consist of the data

- Topological spaces X_i for $0 \le i \le n$
- Maps $f_{i,j} : X_i \times I^{j-i-1} \to X_j$, $0 \le i, j \le n$ such that the 'composition law at $t = 1$' holds for all $0 \le i < j < k \le n$:

$$f_{i,k}\left(-,(u,1,v)\right) = f_{j,k}\left(f_{i,j}\left(-,u\right),v\right)$$

  where the 1 is placed at the $j-i$th component (indexed from 1).

The face map d_k^n are defined by skipping the kth topological space and sets the appropriate skipped element in I^{j-i-1} to 0.
The degeneracy maps s_k^n are defined by duplicating the kth topological space and duplicating the appropriate element in I^{j-i-1}.
One can quickly verify by the definition of a n-morphism, we do indeed get homotopy between homotopies between etc.

Example. Similarly, we can define the category of all ∞ groupoids $\infty\text{-}\mathrm{Gpd}$ as a $(\infty,1)$-category. Furthermore, by defining n-groupoids as the n truncated objects in $\infty\text{-}\mathrm{Gpd}$, we obtain the $(n+1,1)$-category $n\text{-}\mathrm{Gpd}$.

Lemma 1.3.8 (Homotopy hypothesis)**.** *The categories* Top *and* $\infty\text{-}\mathrm{Gpd}$ *are equivalent via the functors* $\Pi : \mathrm{Top} \to \infty\text{-}\mathrm{Gpd}$ *and* $|-| : \infty\text{-}\mathrm{Gpd} \to \mathrm{Top}$

We will also recall the yoneda lemma for $(1,1)$-categories and $(\infty,1)$-categories:

Lemma 1.3.9 ($(1,1)$-Yoneda lemma)**.** *The functor* $\mathcal{C} \to \mathrm{Psh}_{(1,1)}(\mathcal{C}) = \underline{\mathrm{Hom}}_{\mathrm{Cat}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ *is full and faithful.*

Lemma 1.3.10 ($(\infty,1)$-Yoneda lemma)**.** *The functor* $\mathcal{C} \to \mathrm{Psh}_{(\infty,1)}(\mathcal{C}) = \underline{\mathrm{Hom}}_{\infty\text{-}\mathrm{Cat}}(\mathcal{C}^{\mathrm{op}}, \infty\text{-}\mathrm{Gpd})$ *is full and faithful*

And we will see later that covering spaces of a sufficiently nice space X are completely characterized by functors from $\Pi_1(X)$ to Set. This suggests to us that if we want to generalize covering spaces naturally, we should look at functors from $\Pi_n(X)$ to some appropriate generalization of Set to a higher n-category. The goal would be to make this idea more concrete and study how the theory of covering spaces generalizes over. This can be further motivated by considering the n-truncation of the $(\infty,1)$-Yoneda lemma, giving us the functor

$$\mathcal{C} \to \mathrm{Psh}_{(n,1)}(\mathcal{C}) = \underline{\mathrm{Hom}}_{n\text{-}\mathrm{Cat}}(\mathcal{C}^{\mathrm{op}}, n\text{-}\mathrm{Gpd})$$

2 Categorification of covering spaces

We first present an alternative approach to covering spaces via category theory in a way that allows us to generalize the arguments easily. This is a natural setting as one can easily studies the higher homotopy groups via the ∞-groupoid Π of a space. This approach was inspired by tom Dieck's approach to covering spaces [6]. We first provide a rough overview of this section:
Let Cov_X be the subcategory of Top_X where objects are covering spaces. We first aim to construct a functor

$$T : \mathrm{Cov}_X \to \underline{\mathrm{Hom}}_{\mathrm{Cat}}(\Pi_1(X), \mathrm{Set})$$

and the inverse functor for a suitably nice space X

$$T^{-1} : \underline{\mathrm{Hom}}_{\mathrm{Cat}}(\Pi_1(X), \mathrm{Set}) \to \mathrm{Cov}_X$$

Then to study the Galois correspondence, we define the category G - Set that consists of left G-sets and G-equivariant maps for any group G. This leads us to the chain of equivalent categories

$$\mathrm{Cov}_X \overset{T}{\cong} \underline{\mathrm{Hom}}_{\mathrm{Cat}}\left(\Pi_1(X), \mathrm{Set}\right) \overset{R}{\cong} \pi_1(X,x)\text{ - Set}$$

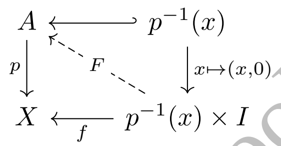
and by restricting ourselves to the subcategory of transitive $\pi_1(X,x)$-sets, these functors give us all the connected coverings, recovering the classical Galois correspondence.

2.1 Construction of $T : \mathrm{Cov}_X \to \underline{\mathrm{Hom}}_{\mathrm{Cat}}\left(\Pi_1(X), \mathrm{Set}\right)$

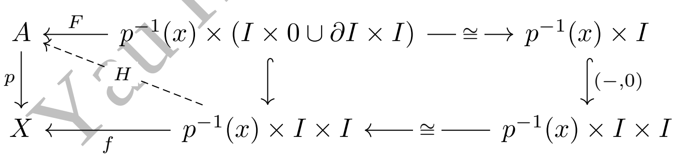
Let $p : A \to X$ be a covering. We first show that this defines a functor $T(p) : \Pi_1(X) \to \mathrm{Set}$ by setting $T(p)(x) = p^{-1}(x)$ and constructing the image of morphisms in $\Pi_1(X)$, then we show that morphisms between coverings give us a natural transformation.

Lemma 2.1.1. *Let $p : A \to X$ be a covering. This naturally defines a functor $T(p) : \Pi_1(X) \to \mathrm{Set}$ with $T(p)(x) = p^{-1}(x)$*

Proof. Given some $f \in \mathrm{Hom}_{\Pi_1(X)}(x,y)$, this comes from some map $f : I \to X$ such that $f(0) = x, f(1) = y$. We need to show that this induces a unique morphism $T(p)(f) : T(p)(x) \to T(p)(y)$. Consider the diagram



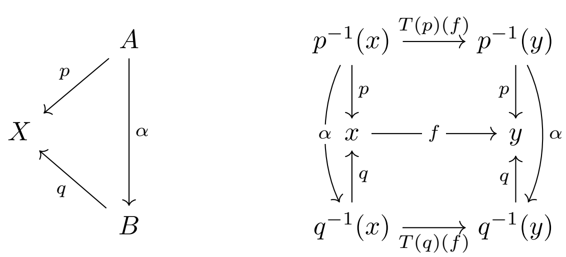
where $f : p^{-1}(x) \times I \to X$ ignores the first component. Since p is a fibration, we know F exists such that the whole diagram commutes. This gives us a morphism from $T(p)(x) \to T(p)(y)$ but it does not gaurentee uniqueness. Given two maps $f_0, f_1 : I \to X$ with a homotopy between them $f : I \times I \to X, f(-,t) = f_-$, since they give us the same morphism in $\Pi_1(B)$, we expect that the liftings F_0, F_1 constructed (note that these liftings are not a priori unique) from the diagram above will give us the same morphism from $T(p)(x) \to T(p)(y)$. This can readily be shown by the diagram



where $F(a,\epsilon,t) = F_\epsilon(a,t), \epsilon \in \{0,1\}$ and $F(-,0,-)$ is constant and f forgets the first component. This gives us a homotopy $F_0(a,1) \to F_1(a,1)$, but since $p^{-1}(x)$ is discrete, this tells us that $F_0(a,1) = F_1(a,1) \in p^{-1}(y)$, hence $f \in \mathrm{Hom}_{\Pi_1(B)}(x,y)$ induces a unqiue morphism $F \in \mathrm{Hom}_{\mathrm{Set}}\left(p^{-1}(x), p^{-1}(y)\right)$. □

Lemma 2.1.2. *Let $p : A \to X, q : B \to X$ be coverings with $\alpha : A \to B$ such that $q\alpha = p$. This gives us a natural transformation $T(\alpha) : T(p) \to T(q)$ by letting α act on the image of $T(p)$.*

Proof. From the condiiton above, we obtain a morphism of fibres by restriction $\alpha_x : p^{-1}(x) \to q^{-1}(x)$. Consider the following diagram



The triangle on the left is by definition of α. The squares on the right commute by definition of $T(-)$ and the outer α of the squares commute by $p = q\alpha$. Hence the whole diagram commutes and we get a natural transformation $T(\alpha)$. □

Theorem 2.1.3. *The previous lemmas define a functor* $T : \mathrm{Cov}_X \to \underline{\mathrm{Hom}}_{\mathrm{Cat}}(\Pi_1(X), \mathrm{Set})$

Proof. We note that the condition in Lemma 2.1.2 is equivalent to a morphism

$$\alpha \in \mathrm{Hom}_{\mathrm{Cov}_X}(p : A \to X, q : B \to X)$$

Hence, combined with Lemma 2.1.1, we have shown that T is indeed a functor. □

2.2 Construction of $T^{-1} : \underline{\mathrm{Hom}}_{\mathrm{Cat}}(\Pi_1(X), \mathrm{Set}) \to \mathrm{Cov}_X$

The main idea to construct the inverse functor is that we can 'glue' trivial fibres on small enough open sets of X in a way compatible with the definition of T. We shall first construct the functor $T^{-1} : \underline{\mathrm{Hom}}_{\mathrm{Cat}}(\Pi_1(X), \mathrm{Set}) \to \mathrm{Cov}_X$, then show that T and T^{-1} are indeed inverses.
We restrict our attention to when X is path-connected, locally path-connected and semilocally simply connected, as without these conditions, the universal cover does not even exist in classical algebraic topology.
We start off by constructing the image of a chosen functor $\Phi : \Pi_1(X) \to \mathrm{Set}$:

Lemma 2.2.1. *The functor* Φ *gives us a covering space.*

Proof. Define $U \in \mathcal{U}$ if U is an open set in X, $\Pi_0(U)$ is trivial and the image of $\mathrm{Hom}_{\Pi_1(U)}(u, u)$ in $\Pi_1(X)$ is trivial.
For every $u \in U \in \mathcal{U}$, $U \times \Phi(u)$ has the obvious topology of a trivial fibre, i.e. we give $\Phi(u)$ the discrete topology,(U,v). To glue all possible choices of u, U together, the appropriate categorical notion to consider is a colimit.
By abuse of notation, let J be a diagram in Top consisting of elements

$$(U \cap V) \times \Phi(u) := (U \cap V, u)\ u \in U \cap V\ U, V \in \mathcal{U}$$

where U, V may be the same element.
We shall now construct such a colimit diagram. The simplest case is a morphism from $(U \cap V, u)$ to (U, u), given by the inclusion map $\iota_{(U\cap V,u),(U,u)}$, which is evidently continuous.
To motivate the most general case, we first construct a morphism from (U, u) to (U, v), where $u, v \in U \in \mathcal{U}$. Let $f \in \mathrm{Hom}_{\Pi_1(U)}(u, v)$ be an arbitrary morphism and define the map

$$\phi_{(U,u),(U,v)}(w, s) = (w, \Phi(f)(s))$$

This is independent of f as $\Phi\left(\mathrm{Hom}_{\Pi_1(U)}(u, v)\right)$ only has a single element by the construction of $\mathcal{U}$, and since it is continuous, this is an appropriate morphism.
Now to construct a morphism from $(U \cap V, u)$ to $(U \cap V, v)$. Intuitively we want to define such a morphism to be the composition

$$(U \cap V, u) \overset{\iota_{(U\cap V,u),(U,u)}}{\to} (U, u) \overset{\phi_{(U,u),(V,v)}}{\to} (U, v) \overset{\iota^{-1}_{(U\cap V,v),(U,v)}}{\to} (U \cap V, v)$$

but we need to verify that the final restriction map $(U, v) \to (U \cap V, v)$ does make sense and that it still makes sense if we swap U, V. However, this is immediate if we select some

$$w \in U \cap V, f \in \mathrm{Hom}_{\Pi_1(U)}(u, w), g \in \mathrm{Hom}_{\Pi_1(V)}(v, w)$$

and define

$$\phi_{(U\cap V,u),(U\cap V,v)}(x, s) = \left(x, \left(\Phi\left(g^{-1}\right)\Phi(f)\right)(s)\right) = \left(x, \Phi\left(g^{-1}f\right)(s)\right)$$

and by similar arguments to above, this is independent of our choice of f, g. Similarly it will be independent if we swap U, V, hence is well defined. To show continuity, notice that for all path connected W such that $x \in W \subseteq U \cap V$, the $\Phi(c)$ component of image of $\phi_{(U\cap V,u),(U\cap V,v)}$ restricted to $W \times \Phi(u)$ is independent of the input from W, and by the locally path connected assumption, this proves continuity. Finally, we consider the space $\operatorname{colim} J$. Every object in J has a natural map to X by inclusion and ignoring the first component. Furthermore, by assumption, X is path-connected, so the fibres are all the same and X is semilocally simply connected, so $\bigcup \mathcal{U} = X$. Finally, every point has a locally trivial neighbourhood by the construction of the colimit. Hence $T^{-1}\Phi : \operatorname{colim} J \to X$ is a covering. □

Lemma 2.2.2. *The above construction gives us the functor* $T^{-1} : \underline{\mathrm{Hom}}_{\mathrm{Cat}}\left(\Pi_1(X), \mathrm{Set}\right) \to \mathrm{Cov}_X$.

Proof. Now we need to verify that it is indeed a functor but this is immediate as a map

$$\alpha \in \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathrm{Cat}}\left(\Pi_(X),\mathrm{Set}\right)}\left(\Phi_1, \Phi_2\right)$$

induces a map

$$(U \cap V) \times \Phi_1(u) \xrightarrow{\alpha} (U \cap V) \times \Phi_2(u)$$

in a natural way, which by passing through the colimit construction in the proof above, gives us that T^{-1} is functorial. □

Theorem 2.2.3. *T is an equivalence of categories.*

Proof. Throughout this proof, we will use the notations introduced in the previous proofs.
We are left with verifying that T and T^{-1} are inverses. We do this by first showing for a covering $p : A \to X$, $T^{-1}(T(p)) \cong p$, then for a functor $\Phi : \Pi_1(X) \to \mathrm{Set}$, $T\left(T^{-1}(\Phi)\right) \cong \Phi$.
Let $T^{-1}(T(p)) = p' : E' \to X$. Recall that Top is a concrete category and since the injection map from each object in the diagram J to E' is a homeomorphism and E' is set theoretically the colimit, $E' \cong E$ as topological spaces by the colimit topology.
Now we prove the final portion of the proof. Define $T^{-1}(\Phi) = p : E \to B$ and let $x \in \Pi_1(X)$, then

$$T\left(T^{-1}(\Phi)\right)(x) = p^{-1}(x) = \Phi(x)$$

by construction. Finally, suppose $x, y \in U \in \mathcal{U}$, then evidently for any $f \in \mathrm{Hom}_{\Pi_1(U)}(x, y)$, we have $T\left(T^{-1}(\Phi)\right)(f) = f$ as there is only one choice of f in $\Pi_1(X)$. Now for any $x, y \in X$ and $f \in \mathrm{Hom}_{\Pi_1(X)}(x, y)$, since $[0, 1]$ is compact, $f([0, 1])$ is compact and there is a finite subcover consisting of elements in $\mathcal{U}$, and since each choice is unique, the composition is unique, giving us $T\left(T^{-1}(\Phi)\right)(f) = f$. □

Corollary 2.2.4. *With the same assumptions as before, $T^{-1}\left(\mathrm{Hom}_{\Pi_1(B)}(b, -)\right)$ for any $b \in B$ is simply connected.*

Proof. Let $X = T^{-1}\left(\mathrm{Hom}_{\Pi_1(B)}(b, -)\right)$, we have the fibre sequence $\pi_1(B) = \mathrm{Hom}_{\Pi_1(B)}(b, b) \to X \to B$. Using the Puppe sequence Theorem 1.1.10, we see that

$$\pi_1\left(\pi_1(B)\right) \cong 0 \to \pi_1(X) \to \pi_1(B) \to \pi_0\left(\pi_1(B)\right) \to 0 \cong \pi_0(X)$$

but since $\pi_1(B) = \pi_0\left(\pi_1(B)\right)$, $\pi_1(X) = 0$ □

2.3 Construction of Galois correspondence

This section aims to construct the equivalence of categories $\underline{\mathrm{Hom}}_{\mathrm{Cat}}\left(\Pi_1(X), \mathrm{Set}\right) \overset{R}{\cong} \pi_1(X, x)\text{-}\mathrm{Set}$. After the tricky constructions in the previous subsections, the construction of the functors R, R^{-1} and showing they are equivalent are relatively simple and does not require messy results from topology. To construct R, we assume that X is path connected so that $\Pi_1(X)$ is connected.

Lemma 2.3.1. *Every functor* $\Phi : \Pi_1(X) \to \mathrm{Set}$ *naturally gives rise to a* $\pi_1(X, x)\text{-}\mathrm{Set}$

Proof. We simply define the set $\Phi(x)$ with the action as $\pi_1(X,x)\times\Phi(x)\to\Phi(x)$ given by $(g,\chi)\to\Phi(g)(\chi)$ using the fact that $\pi_1(X,x)=\operatorname{Hom}_{\Pi_1}(x,x)$. □

Lemma 2.3.2. *The construction above is functorial, giving us the functor R*

Proof. A natural transformation $\alpha:\Phi_1\to\Phi_2$ partly consists of the following data:

$$\begin{array}{ccc} \Phi_1(x) & \xrightarrow{\alpha(x)} & \Phi_2(x) \\ {\scriptstyle \Phi_1(g)}\downarrow & & \downarrow{\scriptstyle \Phi_2(g)} \\ \Phi_1(x) & \xrightarrow[\alpha(x)]{} & \Phi_2(x) \end{array}$$

which is by definition $\pi_1(X,x)$-equivariant, hence we get the functor R. □

Lemma 2.3.3. *Every $\pi_1(X,x)$-set S naturally gives rise to a functor $\Phi:\Pi_1(X)\to\mathrm{Set}$*

Proof. Given any group G and a left G-set S_L and a right G-set S_R, recall the balanced product $S_R\times_G S_L$ is constructed by defining $S_R\times S_L$ as a left G-set with the action $g(r,l)\mapsto\left(rg^{-1},gl\right)$, then construct $S_R\times_G S_L$ by the orbit set of $S_R\times S_L$ under the G-action.
Since $\operatorname{Hom}_{\Pi_1(X)}(x,y)$ is naturally a right $\pi_1(X,x)$ set by composition, we obtain the functor $\Phi(-)=\operatorname{Hom}_{\Pi_1(X)}(x,-)\times_{\pi_1(X,x)}S$ where the maps of morphisms follows by acting on the first component. □

Lemma 2.3.4. *The construction above is functorial, giving us the functor R^{-1}*

Proof. Let $f:S_1\to S_2$ be a $\pi_1(X,x)$-equivariant map and Φ_1,Φ_2 be the constructed functors from the lemma above, then by the definition of equivariant, we obtain a natural transformation from Φ_1 to Φ_2 by acting on the second component. □

Theorem 2.3.5. *R and R^{-1} are inverses*

Proof. Let S be a $\pi_1(X,x)$-set. We have

$$R^{-1}(R(S))=\operatorname{Hom}_{\Pi_1(X)}(x,x)\times_{\pi_1(X,x)}S\cong S\quad(f,s)\mapsto f(s)$$

hence $R^{-1}R$ is the identity.
Let $\Phi:\Pi_1(X)\to\mathrm{Set}$ be a functor. We have

$$R\left(R^{-1}(\Phi)\right)=\operatorname{Hom}_{\Pi_1(X)}((,x),-)\times_{\pi_1(X,x)}\Phi(x)\cong\Phi(-)\quad(f,s)\to\Phi(f)(s)$$

hence RR^{-1} is the identity. □

Corollary 2.3.6. *Suppose X has a universal cover C. The subcategory of connected covers in Cov_X is equivalent to the subcategory of transitive $\pi_1(X,x)$-sets in $\pi_1(X,x)$-Set.*

Proof. Let S be a transitive $\pi_1(X,x)$-set. We note that such a set is equivalent to $\pi_1(X,x)/G$ where G is any subgroup of $\pi_1(X,x)$. We see that

$$R^{-1}(S)=\operatorname{Hom}_{\Pi_1(X)}(x,-)\times_{\pi_1(X,x)}\frac{\pi_1(X,x)}{G}$$

which intuitively 'quotients' the $\Pi_1(X)$. Now consider some connected cover $p:A\to X$ with fibre $T(p)(x)$. We see that the long exact sequence of homotopy groups gives us

$$\pi_1(T(p)(x))\cong 0\to\pi_1(A)\to\pi_1(X)\to\pi_0(T(p)(x))\to 0\cong\pi_0(A)$$

which tells us that $\pi_1(A)$ is a subgroup of $\pi_1(X)$ and $T(p)(x)$ is the coset space $\pi_1(X)/\pi_1(A)$. This immediately shows that the two subcategories are equivalent by associating G with $\pi_1(A)$. □

This final corollary gives us the Galois correspondence between subgroups of π_1 and connected covering spaces promised at the start of this paper and concludes this section.

3 Generalization to higher homotopy groups

Consider the categories $n\text{-}\mathrm{Gpd} = \tau_{\leq n}\infty\text{-}\mathrm{Gpd}$. Define the category $n\text{-}\mathrm{Cov}_X$ as the subcategory of Top_X where maps are fibre bundles with fibres being an element of $(n-1)\text{-}\mathrm{Gpd}$ (after taking the geometric realization), or by using the homotopy hypothesis, the spaces with trivial $\pi_{\geq n}$.

Example. Let $n = 1$, we obtain $n\text{-}\mathrm{Gpd} = \mathrm{Set}$ and $n\text{-}\mathrm{Cov}_X$ being the category of covering spaces.

In this section, we aim to construct the functor

$$T : n\text{-}\mathrm{Cov}_X \to \underline{\mathrm{Hom}}_{\infty\text{-}\mathrm{Gpd}}(\Pi_n(X), (n-1)\text{-}\mathrm{Gpd})$$

for all spaces X, and determine a 'suitably nice condition' for the inverse functor

$$T^{-1} : \underline{\mathrm{Hom}}_{\infty\text{-}\mathrm{Gpd}}(\Pi_n(X), (n-1)\text{-}\mathrm{Gpd}) \to n\text{-}\mathrm{Cov}_X$$

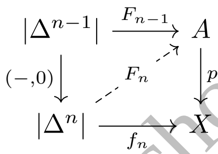
to exist.

3.1 Construction of $T : n\text{-}\mathrm{Cov}_X \to \underline{\mathrm{Hom}}_{\infty\text{-}\mathrm{Gpd}}(\Pi_n(X), (n-1)\text{-}\mathrm{Gpd})$

We shall first construct the functor $T : n\text{-}\mathrm{Cov}_X \to \underline{\mathrm{Hom}}_{\infty\text{-}\mathrm{Gpd}}(\Pi_n(X), (n-1)\text{-}\mathrm{Gpd})$ of a $(n,1)$-categories treated as $(\infty,1)$-categories.

We will require a quick technical lemma

Lemma 3.1.1. *Let $p : A \to X$ be a fibration, then the following lifting problem can be solved:*



Proof. We show this by deforming $|\Delta^{n-1}| \times I$ to $|\Delta^n|$ while keeping $|\Delta^{n-1}| \times \{0\}$ fixed. This can easily be done by simply scaling the rays from the centre appropriately, hence this lifting problem can be solved. ☐

We shall also recall what a n-simplex is in $\infty\text{-}\mathrm{Gpd}$ with some examples:

Definition 3.1.2. *Using the example given in the section on ∞ categories, we have explicitly*

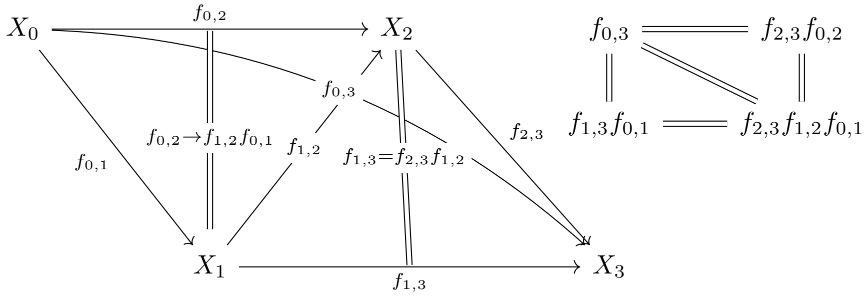
- *A 0-simplex is given by an object X_0*
- *A 1-simplex is given by the 0-simplices X_0, X_1 with a map $f_{0,1} : X_0 \to X_1$*
- *A 2-simplex is given by the 1-simplices $f_{0,1} : X_0 \to X_1, f_{0,2} : X_0 \to X_2, f_{1,2} : X_1 \to X_2$ with a homotopy $H_{0,2} : f_{0,2} \to f_{1,2} \circ f_{0,1}$*
- *A 3-simplex is given by the 2-simplices*

$$\{X_0, X_1, X_2, f_{0,1}, f_{0,2}, f_{1,2}, H_{0,2}\}, \{X_1, X_2, X_3, f_{1,2}, f_{1,3}, f_{2,3}, H_{1,3}\}$$

  and the homotopies

$$H_{0,3,0} : f_{0,3} \to f_{2,3} \circ f_{0,2} \quad H_{0,3,1} : f_{0,3} \to f_{1,3} \circ f_{0,1}$$

The final one is rather hard to understand so we provide a diagram:



In the context of topological spaces, if the morphisms are paths from points, the $\Delta^n - \partial\Delta^n$ would be the higher homotopies.

Note that $\underline{\mathrm{Hom}}_{\infty\text{-}\mathrm{Gpd}}(\Pi(X), (n-1)\text{-}\mathrm{Gpd}) = \underline{\mathrm{Hom}}_{\infty\text{-}\mathrm{Gpd}}(\Pi_n(X), (n-1)\text{-}\mathrm{Gpd})$, hence we shall use the first representation as it is simpler to work with.

Lemma 3.1.3. *Let $p : A \to X \in n\text{-}\mathrm{Cov}_X$, then this defines a functor $T(p) : \Pi(X) \to (n-1)\text{-}\mathrm{Gpd}$ by $T(p)(x) = \Pi\left(p^{-1}(x)\right)$*

Proof. We define $T(p)(x) = \Pi\left(p^{-1}(x)\right)$ for all x as all morphisms are invertible (up to homotopy) in $\Pi(X)$. We note that the geometric realization of this groupoid is the fibre itself by the homotopy hypothesis. Let's recall what it means for $T(p)$ to be a functor in this case. We require for every $f : [b] \to [a]$ in Δ, the following diagram commutes

$$\begin{array}{ccc} \Pi(X)([a]) & \xrightarrow{\Pi(X)(f)} & \Pi(X)([b]) \\ \Phi([a])\downarrow & & \downarrow\Phi([b]) \\ n\text{-}\mathrm{Gpd}([a]) & \xrightarrow[n\text{-}\mathrm{Gpd}([f])]{} & n\text{-}\mathrm{Gpd}([b]) \end{array}$$

With the definition given above, we see that all we need to do is construct a well defined map from n-simplices of X to n-simplices of $p^{-1}(x)$. Let $f_n : |\Delta^n| \to X$ be a n-simplex of X, then we can inductively use the following diagram, which has a lift by Lemma 3.1.1, to construct a n-endomorphism of $T(p)(x)$:

$$\begin{array}{ccc} p^{-1}(x) \times |\Delta^{n-1}| & \xrightarrow{F_{n-1}} & X \\ (-,0)\downarrow & \overset{F_n}{\nearrow} & \downarrow p \\ p^{-1}(x) \times |\Delta^{n}| & \xrightarrow[f_n]{} & A \end{array}$$

which by restricting the domain of F_{n-1} to a n-simplex, we obtain the desired automorphism. Similarly, we need to show that the image of $p^{-1}(x) \times |\Delta^{n-1}|$ is unique, in this case up to n-homotopy as we are mapping into a n-groupoid.
Similar to the previous section, we have a uniqueness of the lifts up to n-homotopy as any n-homotopy acting on $(n-1)$-homotopies in $T(p)$ are trivial. This gives our functor up to n-homotopy which is sufficient. □

Theorem 3.1.4. *The above construction gives us the functor T.*

Proof. Similar to the previous section, the functorality is by definition of a map of n-covers. This gives us the functor T. □

3.2 Construction of $T^{-1} : \underline{\mathrm{Hom}}_{\infty\text{-}\mathrm{Gpd}}(\Pi_n(X), (n-1)\text{-}\mathrm{Gpd}) \to n\text{-}\mathrm{Cov}$

As with the previous section, we first start with a functor $\Phi : \Pi_n(X) \to (n-1)\text{-}\mathrm{Gpd}$ and construct a covering space by gluing fibres. We require a stronger semilocally simply connected condition in this case

Definition 3.2.1. *X is n-semilocally simply connected if for every $x \in X$, there exists a $x \in U$ such that $\Pi_0(U)$ is trivial and the image of $\mathrm{Hom}_{\Pi_n(U)}(u, u)$ in $\Pi_n(X)$ is trivial.*

Similar to the classical case, we assume that X is path-connected, locally path-connected, n-semilocally simply connected in order to construct the functor T^{-1}.

Lemma 3.2.2. *The functor Φ gives us a covering space.*

Proof. Define $U \in \mathcal{U}$ if U is an open set in X, $\Pi_0(U)$ is trivial and the image of $\mathrm{Hom}_{\Pi_n(U)}(u, u)$ in $\Pi_n(X)$ is trivial.

For every $u \in U \in \mathcal{U}$, $U \times |\Phi(u)|$ has the obvious topology of a trivial fibre. To glue all possible choices of u, U together, we do the same construction as in the previous proof for the classical case.
By abuse of notation, let J be a diagram in Top consisting of elements

$$(U \cap V) \times |\Phi(u)| := (U \cap V, u) \ u \in U \cap V \ U, V \in \mathcal{U}$$

where U, V may be the same element.
Using a similar construction as the previous proof, we construct the maps $\phi_{(U\cap V,u),(U\cap V,v)}$ in the diagram using the 1-morphisms.
We can immediately extend this to the k-morphisms, $k \leq n+1$, by mapping k-simplices in $\Phi(u)$ from $(U \cap V, u)$ to $(U \cap V, v)$. This is enabled with the modified condition that the image of $\mathrm{Hom}_{\Pi_n(U)}(u, u)$ in $\Pi_n(X)$ is trivial.
Finally, we consider the space $\operatorname{colim} J$. Every object in J has a natural map to X by inclusion and ignoring the first component. Furthermore, by assumption, X is path-connected, so the fibres are all the same and X is n-semilocally simply connected, so $\bigcup \mathcal{U} = X$. Finally, every point has a locally trivial neighbourhood by construction of the colimit. Hence $T^{-1}\Phi : \operatorname{colim} J \to X$ is a covering. □

Theorem 3.2.3. *T is an equivalence of categories*

Proof. The proof here is extremely similar. The fact that $T^{-1}T$ is the identity comes from the fact that Top is a concrete category and for any Φ, we can show $TT^{-1}(\Phi) \cong \Phi$ up to a n-homotopy by using the fact that $[0,1]^n$ is compact and X is n-semilocally simply connected in an identical manner as the previous section. □

3.3 Construction of Galois correspondence

Similar to the previous section, we will assume X is path-connected. To construct our Galois correspondence, we will need a generalization of $\pi_1(X, x)$ - Set. We see that we can redefine $\pi_1(X, x)$ - Set as functor category of functors from $\mathrm{Hom}_{Pi_1(X)}(x, x)$ to Set. This suggests the proper generalization is by considering the full subcategory $\Pi_n(X)|_x$ of $\Pi_n(X)$ that only consists of the one object x, then the category $\underline{\mathrm{Hom}}_{n\text{-}\mathrm{Cat}}(\Pi_n(X)|_x, (n-1)\text{-}\mathrm{Gpd})$ as a substitute for $\pi_1(X, x)$ - Set, which reduces down to the simpler case for $n = 1$.
In this perspective, the constructions of the functor

$$\underline{\mathrm{Hom}}_{\infty\text{-}\mathrm{Gpd}}(\Pi_n(X), (n-1)\text{-}\mathrm{Gpd}) \overset{R}{\rightleftharpoons} \underline{\mathrm{Hom}}_{n\text{-}\mathrm{Cat}}(\Pi_n(X)|_x, (n-1)\text{-}\mathrm{Gpd})$$

is somewhat easier than the previous section as it is simply stating that the categories $\Pi_n(X)$ and $\Pi_n(X)|_x$ are equivalent induces by the diagonal functor $\Delta_x : \Pi_n(x) \to \Pi_n(X)|_x$ and the inclusion $\iota : \Pi_n(X)|_x \hookrightarrow \Pi_n(X)$ as $\Pi_n(X)$ is connected.
To get a reasonable Galois correspondence here, we need to consider what functors in

$$\underline{\mathrm{Hom}}_{n\text{-}\mathrm{Cat}}(\Pi_n(X)|_x, (n-1)\text{-}\mathrm{Gpd})$$

correspond to 'subgroups'. This is precisely given by the subcategory of full functors. In this case, we will also get that the n-morphisms of the image are precisely subgroups of $\pi_n(X, x)$.
This gives us our Galois correspondence from subgroups to covering spaces, namely:

Theorem 3.3.1. *We have the functors*

$$n\text{-}\mathrm{Cov}_x \overset{T}{\rightleftharpoons} \underline{\mathrm{Hom}}_{\infty\text{-}\mathrm{Gpd}}(\Pi_n(X), (n-1)\text{-}\mathrm{Gpd}) \overset{R}{\rightleftharpoons} \underline{\mathrm{Hom}}_{n\text{-}\mathrm{Cat}}(\Pi_n(X)|_x, (n-1)\text{-}\mathrm{Gpd})$$

if X is path-connected, simply path-connected and n-semilocally simply connected.

Proof. The functor R was constructed in the discussion above and T in the previous subsections. □

And as before, we can define the universal cover as $C_{n,X} = T^{-1}\left(R^{-1}\left(\Phi_{\text{id}}\right)\right)$ where Φ_{id} is the functor that sends everything into the trivial groupoid of one object and only the identity, then we have

Corollary 3.3.2. *Furthermore, the subcategory of connected covers is equivalent to the subcategory of full functors under the functor $R \circ T$. This tells us that $T^{-1}R^{-1}$ maps certain tuples of subgroups of $\pi_{\leq n}(X, x)$ to connected n-covers.*

Proof. We have similar to the previous section that these are simply quotients of the universal cover. The fact that we do get a nice subgroup structure is a direct application of the Puppe sequence. The image of R^{-1} gives us the fibre, which when put under into the Puppe sequence, gives us a connected cover with a exact sequence giving us the homotopy groups of the cover. Let the fibre and cover formed from this be $F = \left|R^{-1}(-)\right|$ and $A = T^{-1}\left(R^{-1}(-)\right)$ respectively, we have

$$\pi_n(F) = 0 \to \pi_n(A) \to \pi_n(X) \to \pi_{n-1}(F) \to \cdots \to \pi_1(A) \to \pi_1(X) \to \pi_0(F) \to 0 \cong \pi_0(A)$$

□

This final corollary gives us our Galois correspondence between subgroups of $\pi_{\leq n}$ and connected n-covering spaces.

4 Conclusion and future work

We have proven usual facts about the universal cover, which exists over a connected, simply-connected, semilocally simply-connected space. In contrast to usual topological construction, we used a categorical approach here, leaning towards infinity categories and conclude by proving the Galois correspondence between connected covers and subgroups by constructing the functors

$$\text{Cov}_X \overset{T}{\cong} \underline{\text{Hom}}_{\text{Cat}}\left(\Pi_1(X), \text{Set}\right) \overset{R}{\cong} \pi_1(X, x)\text{-Set}$$

from the category of covering spaces to an intermediate functor category then to the category of all sets with a left $\pi_1(X, x)$ action. This gives us the universal cover as the preimages of the singleton set in $pi_1(X, x)\text{-Set}$.

This lets us port everything over into higher categories and generalize all our theorems by replacing this with

$$n\text{-Cov}_X \overset{T}{\cong} \underline{\text{Hom}}_{n\text{-Gpd}}\left(\Pi_n(X), (n-1)\text{-Gpd}\right) \overset{R}{\cong} \underline{\text{Hom}}_{n\text{-Gpd}}\left(\Pi_n(X)|_x, (n-1)\text{-Gpd}\right)$$

for path-connected, locally path-connected and n-semilocally simply connected spaces where n-semilocally simply connected is a stronger condition that requires vanishing π_n for covers.

In this context, our universal cover is the preimage of the trivial functor, namely the one that sends objects to the trivial groupoid consisting of only one object and only the identity. This map also maps certain tuples of subgroups of $\pi_{\leq n}(X, x)$ to connected n-covers, which gives us our Galois correspondence where the subgroups are now 'subgroupoids' to ensure that all the homotopy groups are compatible, for instance in the Puppe sequence, to form a connected n-cover.

In the future, we would like to provide a more concrete condition on the subgroups that can be phrased purely group-theoretically as well as study the universal n-cover in more concrete contexts. Possible appearances include the study of higher Lie groups from string theory [15]. One could also hope that we can modify this to work with various types of manifolds and then extend it to complex manifolds and finally algebraic geometry.

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