2021 S.T. Yau High School Science Award (Asia)

Research Report

The Team
Registration Number: MA-072

Name of team member: Zhou Kangyun
School: Hwa Chong Institution
City, Country: Singapore, Singapore

Name of supervising teacher: Chenglong Yu
Job Title: Assistant Professor
School/Institution: Tsinghua University
City, Country: Beijing, China

Title of Research Report: Padé approximations in Diophantine Approximations and Irrationality Problems about Confluent Hypergeometric Functions

Date: 2021/08/31

Padé approximations in Diophantine Approximations and Irrationality Problems about Confluent Hypergeometric Functions

Zhou Kangyun

Abstract

Padé approximations are approximations of holomorphic functions by rational functions. The application of Padé approximations to Diophantine approximations has a long history dating back to Hermite. In this paper, we use Maier-Chudnovsky construction of Padé-type approximation to study irrationality properties of functions with the form

$$f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!(bk+s)(bk+s+1)...(bk+t)},$$

where b, t, s are positive integers and $g.c.d(b, t) = 1$ and obtain upper bounds for irrationality measures of their values at non-zero rational points.

Keywords: Padé approximation, Diophantine approximation, irrationality, confluent hypergeometric function

Acknowledgement.

The author would like to thank the supervising teacher Chenglong Yu for his help with the project and introducing the author to the beautiful branch of number theory, Diophantine approximations.

The author would also like to thank his friends and parents for all the support and encouragement during the period of his research.

Commitments on Academic Honesty and Integrity

We hereby declare that we

1. are fully committed to the principle of honesty, integrity and fair play throughout the competition.
2. actually perform the research work ourselves and thus truly understand the content of the work.
3. observe the common standard of academic integrity adopted by most journals and degree theses.
4. have declared all the assistance and contribution we have received from any personnel, agency, institution, etc. for the research work.
5. undertake to avoid getting in touch with assessment panel members in a way that may lead to direct or indirect conflict of interest.
6. undertake to avoid any interaction with assessment panel members that would undermine the neutrality of the panel member and fairness of the assessment process.
7. observe the safety regulations of the laboratory(ies) where we conduct the experiment(s), if applicable.
8. observe all rules and regulations of the competition.
9. agree that the decision of YHSA(Asia) is final in all matters related to the competition.

We understand and agree that failure to honour the above commitments may lead to disqualification from the competition and/or removal of reward, if applicable; that any unethical deeds, if found, will be disclosed to the school principal of team member(s) and relevant parties if deemed necessary; and that the decision of YHSA(Asia) is final and no appeal will be accepted.

(Signatures of full team below)

周康耀

Name of team member: Zhou Kangyun

Name of team member:

Name of team member:

余成龙

Name of supervising teacher: Cheng Long Yu

Noted and endorsed by

(signature) 1/9/2021

Name of school principal: CHIN BEE KUAN (MRS)
DEPUTY PRINCIPAL
HWA CHONG INSTITUTION
SINGAPORE

Contents

1 Introduction

Ever since Hippasus of Metapontum made the astonishing discovery of the existence of irrational number, the irrationality or transcendence of special values has been attracted much attention. From the simplistic proof for the irrationality of $\sqrt{2}$ and e, mathematicians have developed a variety of different measures to prove the irrationality of special values, as well as estimation its irrationality in terms of irrationality measure.

The French mathematician, Charles Hermite, first came up with the idea of using rational function to approximate and prove the irrationality or transcendence of values of exponential function at rational points. This method of approximation by rational functions is called Padé approximation. This method motivated many mathematicians to study Diophantine approximations, for example, Maier [4], Siegel [6], Chudnovsky [2], etc.

One of the most memorable and significant, yet surprising finding in irrationality of values would be Apéry's proof of $\zeta(3)$'s irrationality in 1978. In the original sketch of proof published by Apéry involves two novel series A_n and B_n named Apéry numbers and use of their quotients to approach the value of $\zeta(3)$ (See [7]), which had once been considered miraculous and unexpected.

However, such series A_n and B_n for many other Diophantine approximation problems naturally appear in Padé approximation or Padé type approximation, especially in the study of generalised hypergeometric functions. A number of significant works have been done by Maier [4], followed by Chudnovsky [2], who have sketched a general form of Padé approximants to generalised hypergeometric functions.

In this paper, we will follow the idea of using Padé or partial Padé approximants to study the irrationality of values of special functions. We will first recover the result of irrationality proof of the exponential values at rational points, and then study on the irrationality property of cumulative distribution function (CDF) of normal distribution. Finally, we will study the irrationality property of some special types of generalised hypergeometric functions, construct the Maier-Chudnovsky type approximations and derive upper bounds for the irrationality measures. These functions include exponential functions, CDF of normal distribution and confluent functions with some integral or rational parameters.

In section 2, we review some basic background including Taylor expansion, Padé approximation and irrationality measure. In section 3, we recall the Padé approximation of exponential function from the differential-operator point of view by Siegel, and also from a more combinatorial lemma used by Maier and Chudnovsky. This also leads to precise estimations of denominators and remainders. In section 4, we construct explicit Padé-type approximants to CDF of normal distribution and prove irrationality via a mod p argument similar as Maier [4]. In section 5, we study the irrationality property of a special form of confluent hypergeometric functions.

2 Background

2.1 Taylor expansion and irrationality of e

Taylor series is one of the earliest rational approximation to functions. Taking the first k terms of the Taylor series, Taylor polynomials is widely used to approximate functions that are infinitely differentiable at some points, especially holomorphic functions. Using Taylor expansion and Taylor polynomial,a proof of irrationality of e is easily obtained. (See [6])

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

$$f_n = \sum_{k=0}^{n} \frac{1}{k!}; \quad R_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}.$$

Notice that

$$R_n = \frac{1}{(n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)}...) < \frac{e-1}{(n+1)!}$$

Therefore

$$0 < n!R_n < \frac{2}{n+1} < 1$$

Which implies that $n!e$ is never an integer, i.e. e is irrational.

Taylor expansion, however, fails to prove the irrationality of e^x when x is a positive integer greater than 1, since the remainder terms do not approach 0 after multiplying $n!$ in this case. A more accurate rational approximation method is required to produce smaller remainder terms.

2.2 Padé Approximation

Padé approximation is introduced to provide a better approximation with larger domain of convergence.

Padé Approximation is usually defined by:

$$f_{[n,m]}(x) = \frac{B_m(x)}{A_n(x)}, x \to a$$

where $B_m(x)$ is polynomial of degree m and $A_n(x)$ is monic polynomial of degree n, and

$$f(a) = f_{[n,m]}(x)$$

$$f'(a) = f'_{[n,m]}(x)$$

$$f''(a) = f''_{[n,m]}(x)$$

$$\cdots$$

$$f^{(m+n)}(a) = f_{[n,m]}^{(m+n)}(x)$$

or alternatively, Padé approximants of f could be defined as:

$$f(x) - \frac{B_m(x)}{A_n(x)} = O((x-a)^{n+m}), x \to a$$

or:

$$A_n(x)f(x) - B_m(x) = O((x-a)^{n+m}), x \to a$$

if we linearise this condition.

2.3 Irrationality proof by Padé approximants

Using Padé approximation, the proof of irrationality to specific values of a function can be directly derived by a non-vanishing remainder term $R(x)$ that approaches 0 fast enough as m, n approaches infinity. This will directly lead to the condition:
For $\alpha = f(x)$ with x fixed, there exists infinitely many pairs of integers (p_n, q_n) such that

$$0 < |q_n\alpha - p_n| \to 0.$$

Based on the previous idea, Maier has investigated on the irrationality of some special generalised hypergeometric functions using partial Padé approximation, which scarifies accuracy of the approximation order but still gives an irrationality proof.

Inspired by Maier's work, Siegel has studied a more general class of functions: Type-E functions. He has proven the irrationality of Type-E functions which satisfies "normal condition", which has been removed by Shidlovsky in his later works. This results has been generally cited as "Siegel-Shidlovsky Theorem", which was later refined by F.Beukers (See [1]). This theorem requires the verification of $Q(x)$-linear independence of some Type-E functions as solutions to some ODE together with their derivatives, which involves differential Galois theory.

Even though this general theorem already gives irrationality results if we know $Q(x)$-linear independence results of these functions, explicit Padé-type approximation can give more information, for example, bounds for irrationality measures.

2.4 Irrationality measure

First we recall the definition of irrationality measure (or irrationality exponent).

Definition 2.1. *Let α be a real number. The irrationality measure $\mu(\alpha)$ is the largest possible value for μ such that*

$$0 < |x - \frac{p}{q}| < \frac{1}{q^\mu}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

Beside the effort on proving the irrationality of unknown function, irrationality measure (or irrationality exponent, in some literature) of known irrational function values has also attracted much attention. Many has joined the competition to get better bounds or accurate values for irrationality measures of important numbers. One of the most famous example would be the result about bounds of irrationality measure of π given by Salikhov, who reduced the upper bound to 7.606308... (See [5]).

In this paper, we will study a type of functions with similar form as confluent hypergeometric functions with rational parameters and generalise Maier's results following his methods, including some interesting examples such as cumulative distribution function of normal distribution. By explicit construction of Padé or partial Padé approximants to the functions, we will also derive an upper bound for the irrationality measures of the rational values of these functions.

3 Irrationality of e^x

Irrationality proof to the exponential function is well-studied in the history. The [n,n] Padé approximation result has been formally given by Siegel, who used differential operator D (differentiation with respect to x) to express $A(x)$ and $B(x)$.

3.1 Siegel's method of differential operators

One of the most classic work of proving the irrationality of special values of a function is the proof of the irrationality of e^x. We will go first recall Siegel's method. (See [6])

Constructing a $[n,n]$ Padé approximation to the function e^x, Siegel solved the equation by using the differential operator $D = \frac{d}{dx}$ for multiple times.

$$A(x)e^x - B(x) = R(x)$$
$$A(x) = (D+1)^{-n-1}x^n;$$
$$B(x) = (D-1)^{-n-1}x^n;$$

Here $(D+1)^{-n-1}$ is understood as Taylor expansion of D at 0. It operates on polynomials and the summation only has finitely many non-zero terms. So is $(D-1)^{-n-1}$. And this implies that both $A(x)$ and $B(x)$ have integer coefficients. And an integral representation of the remainder terms is given by:

$$R(x) = \frac{x^{2n+1}}{n!}\int_0^1 t^n(1-t)^n e^{tx}dt.$$

This representation of $R(x)$ implies that:

$$0 < |R(x)| \le \frac{|x|^{2n+1}e^{|x|}}{n!}.$$

The irrationality of e^x for all rational $x \neq 0$ is proven.

3.2 [n,m] Padé approximants of e^x

In this section, we will derive the $[n, m]$ padé approximants for e^x using a different approach. For the very general form of Padé approximation:

$$A(x)f(x) - B(x) = R(x)$$

where $A(x), B(x)$ are polynomials with degree of n, m; $R(x) \sim O(x^{m+n+1})$ (i.e. the term of lowest order of x in $R(x)$ is order $m + n + 1$).

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

We aim to construct $A(x) = \sum_{i=0}^{\infty} a_i x^i$ with the property that $A(x)f(x) = \sum_{k=0}^{\infty} E_k x^k$, $E_k = 0$ for $m + 1 \leq k \leq m + n$. We shall now introduce an important lemma that we will be using frequently in the rest parts of the paper to construct an explicit form of Padé approximants $A(x), B(x)$.

Lemma 3.1. *For integer $k < n$,*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i i^k = 0$$

Proof. Take the differential operator $\delta = x\frac{\sigma}{\sigma x}$. Notice that

$$(\delta)^k x^i = (i)^k x^i.$$

$$\begin{aligned} \sum_{i=0}^{\infty} \binom{n}{i} (-1)^i i^k &= \sum_{i=0}^{\infty} \binom{n}{i} (-1)^i (\delta)^k x^i \Big|_{x=1} \\ &= (\delta)^k (1-x)^n \Big|_{x=1} \\ &= 0 \end{aligned}$$

□

By addition of terms, this lemma directly leads to:

$$\sum_{i=0}^{\infty} \binom{n}{i} (-1)^i S(i) = 0, ord(S) < n$$

The lemma has been used by Maier and later Chudnovsky to construct explicit forms of Padé approximants.

This lemma inspires us to construct $A(x)$ in such a way that its numerator will cancel with the denominator of $f(x)$.

We can construct an explicit form of $A(x)$:

$$A(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (m+1)_i x^{n-i}$$

Which satisfies the conditions above.

$$A(x)f(x) = \sum_{k=0}^{\infty} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{1}{m!} \frac{(m+i)!}{k!} x^{k+n-i} = \sum_{k=0}^{\infty} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{1}{m!} \frac{(m+i)!}{(k-n+i)!} x^k$$

$$B(x) + \sum_{k=n}^{m+n-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{1}{m!} \frac{(m+i)!}{(k-n+i)!} x^k + R(x)$$

$$\frac{(m+i)!}{(k-n+i)!} = (i+m)(i+m-1)...(i+k-n+1)$$

We need it to be a polynomial of degree $ord < n$ with respect to i. Hence, we will need $0 \leq m - (k-n+1) + 1 < n$, which leads to $m \leq k \leq m+n-1$.

$$\sum_{k=m}^{n+m-1} (-1)^i \binom{n}{i} \frac{1}{m!} \frac{(m+i)!}{(k-n+i)!} x^k = 0$$

This will give us the explicit form of $R(x)$ and $B_m(x)$:

$$R(x) = \sum_{k=m+n}^{\infty} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{1}{m!(m+i+1)...(k-n+i+1)} x^k$$

$$B(x) = \sum_{k=0}^{m-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(m+1)_i}{(k-n+i)!} x^k$$

By the shift of $k \to m+n+k$, we may rewrite $R(x)$ in a better form:

$$R(x) = \frac{x^{m+n}}{m!} \sum_{k=0}^{\infty} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{x^k}{(m+i+1)_k} = \frac{x^{m+n}}{m!} \sum_{k=1}^{\infty} \sum_{i=0}^{n} (-1)^i \binom{n}{i} {}_1F_1 \left[\begin{matrix} 1 \\ m+i+1 \end{matrix} ; x \right]$$

We can derive an integral representation for the hypergeometric function by solving related ODE, which will help us to estimate the growth rate of the remainder term $R(x)$ with respect to m, n and obtain an irrationality proof for e^x.

For the sake of convenience, we write $m+i+1 = a$.

The hypergeometric function ${}_1F_1\left[{1 \atop a+1};x\right]$ satisfies the second-order ODE :

$$(\delta(\delta+a)-x(\delta+1))f(x)=0.$$

By transformation:

$$f^{'}+\frac{a-x}{x}f=\frac{c_1}{x}.$$

Where c_1 is a constant. We multiply the integration factor $e^{-x}x^a$ on both sides of the equation to get the equation:

$$(e^{-x}x^af)^{'}=c_1x^{a-1}e^{-x}$$

$$f=c_1x^{-a}e^x\int_0^x t^{a-1}e^{-t}dt+c_2x^{-a}e^x$$

We will derive an explicit form of $f={}_1F_1\left[{1 \atop a+1};x\right]$ by taking the extreme $x\to 0$.

$$\lim_{x\to 0}{}_1F_1\left[\begin{matrix}1\\a+1\end{matrix};x\right]=1$$

$$\lim_{x\to 0}c_1x^{-a}e^x\int_0^x t^{a-1}e^{-t}dt=\lim_{x\to 0}\frac{x^{a-1}}{ax^{a-1}}=1$$

This directly gives $c_1=a$. Hence, ${}_1F_1\left[{1 \atop a+1};x\right]=ax^{-a}e^x\int_0^x t^{a-1}e^{-t}dt$. We rewrite the equation with $u=\frac{x}{t}$.

$${}_1F_1\left[\begin{matrix}1\\a+1\end{matrix};x\right]=ae^x\int_0^1 u^{a-1}e^{-ux}du=1+xe^x\int_0^1 u^ae^{-ux}du$$

$$R(x)=\frac{x^{m+n}}{m!}\sum_{k=1}^{\infty}(-1)^i\binom{n}{i}(xe^x\int_0^1 u^{m+i}e^{-ux}du)=\frac{x^{m+n+1}}{m!}(\int_0^1 u^m(1-u)^ne^{(1-u)x}du)$$

This result provides the $[m,n]$ Padé approximation result to e^x. By taking $m=n$, Siegel's result of $[n,n]$ (orthogonal) Padé approximation to e^x is fully recovered.

3.3 Irrationality measure of e^x

We have the following integral representation for $A(x)$ from the expansion formula above:

$$\begin{aligned}
A(x) &= \sum_{i=0}^{n}(-1)^i\tbinom{n}{i}\frac{\Gamma(m+i+1)}{m!}x^{n-i}\\
&= \frac{x^{n-i}}{m!}\sum_{i=0}^{n}(-1)^i\tbinom{n}{i}\int_0^{\infty}t^{m+i}e^{-t}dt\\
&= \frac{1}{m!}\int_0^{\infty}t^m(x-t)^ne^{-t}dt\\
&= \frac{1}{m!}\int_0^{x}t^m(x-t)^ne^{-t}dt+\frac{1}{m!}\int_x^{\infty}t^m(x-t)^ne^{-t}dt
\end{aligned}$$

When n is large enough, we will always have

$$\int_x^\infty (t-x)^{n+m}e^{-t}dt < \int_x^\infty t^m(x-t)^n e^{-t}dt < \int_x^\infty t^{m+n}e^{-t}dt \sim \Gamma(m+n+1)$$

We take $m=n$, the growth rate of $A(x)$ is given by $\frac{(2n)!}{n!}$.

Now we estimate the remainder term:

$$R(x) = \frac{x^{2n+1}}{n!}(\int_0^1 (u-u^2)^n e^{(1-u)x}du)$$

The factor $e^{(1-u)x}$ could be neglected as it gives a factor not related to n.
Using trigonometric substitution $u = sin^2 s$:

$$\begin{aligned}\int_0^1 (u-u^2)^n du &= \int_0^{\frac{\pi}{2}} (\sin^2 s\cos^2 s)^n \, d\sin s\cos s \\ &= \frac{(2n)!!}{(2n+1)!!2^{2n}}\end{aligned}$$

So $R(x) \sim x^{2n+1}\cdot\frac{n!}{(2n+1)!}$. Hence, $A(x)R(x) \sim O(\frac{x^{2n+1}}{2n+1})$. When $x=\frac{p}{q}$ with p and q non-zero integers, we need to multiply q^n on both sides of $A(x)e^x - B(x) = R(x)$ and we still have the following inequality for n large enough

$$(q^n R(x)) < \frac{1}{(q^n A(x))^{1-\epsilon}}$$

with any $\epsilon > 0$ fixed. This implies the irrationality measure of $\mu(e^x) = 2$. More generally, the precise estimates above helped Davis to obtain the following.

Theorem 3.2 (Davis, [3]). *For any $\epsilon > 0$, there exists an infinite sequence of rational numbers $\frac{p}{q}$ such that*

$$\left|e - \frac{p}{q}\right| < \left(\frac{1}{2}+\varepsilon\right)\frac{\ln\ln q}{q^2\ln q}$$

The constant $\frac{1}{2}$ is not improvable.

4 Irrationality Property of CDF of Normal Distribution

In this section, we will prove the irrationality as well as find an upper bound for the irrationality measure of the function

$$f(x) = \sum_{k=0}^{\infty}\frac{x^k}{(2k+1)k!}.$$

It is related to CDF of normal distribution function $\Phi(x)$ via the following

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{2} + \frac{x}{\sqrt{2\pi}} f(-\frac{x^2}{2})$$

We will first construct an explicit form of Padé approximants to the function $f(x)$, and then estimate the growth rate of the remainder term to prove that it approaches 0 for infinite number of approximants. Next, we will use a p-adic measure to prove the non-vanishing of the remainder terms to complete the proof of irrationality.

4.1 Padé approximants and estimation of remainder terms

We will first briefly introduce similar work done by Maier.

Historically, in Maier's original paper, he has provided the proof for irrationality of a similar function $\zeta(q)$:

$$\zeta(q) = \sum_{k=0}^{\infty} \frac{q^k}{k!k}$$

His construction of Padé approximants was complicated as it involves differential operators of two variables. (See [4]) With lemma 3.1, however, we can directly construct an explicit form of Padé approximants for this kind of functions.

The form of (partial) Padé approximation is given by:

$$A(x)f(x) - B(x) = R(x).$$

To eliminate the factor $k!$ in the denominator of $f(x)$, the factor $\frac{(m+i)!}{(i)!}$ in $A(x)$ is needed (similar to the previous section); to eliminate the factor $(2k+1)$ in denominator of $f(x)$, we will put a $\binom{2m+2i+1}{2m}$ in $A(x)$, as it will produce factors $(2i+2)(2i+3)...(2i+2m+1)$ in the numerator, which will cancel the term $2(k-n+i)+1$ in the denominator of $A(x)f(x)$ for a range of k. An explicit form of $A(x)$:

$$A(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(m+i)!(2m+2i+1)!}{m!(2i+1)!(2m)!} x^{n-i}$$

So that:

$$A(x)f(x) = \sum_{k=0}^{\infty} E_k x^k = \sum_{k=0}^{\infty} \sum_{i=max(0,n-k+1)}^{n} (-1)^n \binom{n}{i} \frac{1}{m!(2m)!} \frac{(m+i)!(2i+2)...(2i+2m+1)}{(k-n+i)!(2i+2k-2n+1)} x^k$$

When $max(3m-1, n+1) \le k \le m+n$, $E_k = 0$.

$$B(x) = \sum_{k=0}^{max(n-1,3m-1)} \sum_{i=n-k}^{n} (-1)^i \binom{n}{i} \binom{m+i}{m} \binom{2m+2i+1}{2m} \frac{i!}{(k-n+i)!(2i+2k-2n+1)} x^k$$

$$R(x) = \frac{x^{m+n}}{m!} \sum_{k=0}^{\infty} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{2m+2i+1}{2m} \frac{(m+i)!}{(k+m+i)!(2i+2k+2m+1)} x^k$$

When $m, n \to \infty$:

$$R(x) \sim o(\frac{1}{(m!)^{(1-\epsilon)}}) \to 0.$$

However, this form of remainder terms does not inherently state its non-zero property (unlike e^x), as the factor $(-1)^i$ exists so that the sign for every term is not determined.

4.2 Non-vanishing of remainder terms

In this section, we will use a number theory measure to prove the non-vanishing of the remainder terms, by showing that it is non-zero $\mod p$, where p is a prime number.

This method was also originally from Maier's paper (See [4]), who constructed an special infinite series of m, n based on prime numbers, so that the approximant $B(x)$ has a non-zero value $\mod p$ while $p | A(x)$. This will directly lead to the non-zero property of the remainder term if we suppose $f(x)$ is rational.

Suppose $f(x) = \frac{v}{u}$ (assume $f(x)$ is rational, otherwise we have already arrived at the irrationality result), where $v, u = 1, 2, ...$ Let $d = l.c.m[1, 3, ..., 2n-1]$. Notice that $dB(x)$ is an integer, as d will eliminate all terms $(2i+2k-2n+1)$ in the denominator, and $dA(x)$ is obviously an integer. We aim to prove that for fixed x, there are infinite sets of m, n that $p \not| dB(x)$ while $p | A(x)$. If so, for sufficiently large p:

$$dA(x)f(x) - dB(x) = dR(x)$$

$$udA(x)v - udB(x) = udR(x) \not\equiv 0 \mod p.$$

After trials and experiment, we have come up with an infinite sequence of m, n that satisfies the above condition. Our motive is to let most of the terms in $B(x)$ divisible by p, so that we can easily prove the sum of few terms left is still not divisible by p.

$$m = \frac{p-3}{2}$$

$$n = \frac{3p+1}{2}$$

For the sake of convenience, we denote

$$d\binom{n}{i}\binom{m+i}{m}\binom{2m+2i+1}{2m}\frac{i!}{(k-n+i)!(2i+2k-2n+1)}x^k$$

as $B_{(i,k)}(x)$.

$$p \not| dB(x)$$

When

$$m = \frac{p-3}{2}, n = \frac{3p+1}{2}$$

.

Proof. We will analyse the value of $B_{(i,k)}(x)$ based on values of i. Firstly, notice the bound for i, k: $i \leq n, k \leq n-1, 2(k-n+i)+1 \leq 2n-1 = 3p$. We consider the following different conditions:

1. When $p \not| 2i+2k-2n+1$: Since $p|d$, we have

$$p | \frac{d}{(2(k-n+i)+1};$$

   So

$$p | B_{(i,k)}(x).$$

2. When $p|2i+2k-2n+1$: Notice that $v_p(d) = 1$, so

$$p \not| \frac{d}{2(k-n+i)+1}$$

   We consider the following subcases,

   (a) When $2(k-n+i)+1 = p$:

      i. When $i \geq p$, Since $k-n+i = \frac{2p-1}{2}$, obviously, we have

$$p | \frac{i!}{(k-n+i)!};$$

      and

$$p | B_{(i,k)}(x).$$

      ii. When $\frac{p+1}{2} + 1 \leq i < p$, we have

$$\binom{n}{i} = \frac{p + \frac{p+1}{2}}{i!(p + \frac{p+1}{2} - i)!},$$

      As $p + \frac{p+1}{2} - i < p$, $v_p(i!(p + \frac{p+1}{2} - i)!) = 0$; $v_p(p + \frac{p+1}{2}) = 1$, so

$$p | \binom{n}{i}$$

      .

      iii. When $i = \frac{p+1}{2}, k = \frac{3p-1}{2}$, we have

$$B_{(k,i)}(x) = \frac{d}{p}(-1)^{\frac{p+1}{2}} \binom{\frac{3p+1}{2}}{\frac{p+1}{2}} \binom{p-1}{\frac{p+1}{2}} \binom{2p-1}{p-3} \frac{p+1}{2} x^{\frac{3p-1}{2}}.$$

(b) When $2(k-n+i)+1=3p$, we have

$$i=n=\frac{3p+1}{2}; k=n-1=\frac{3p-1}{2},$$

$$B_{(k,i)}(x)=(-1)^{\frac{3p+1}{2}}\frac{d}{3p}\binom{2p-1}{\frac{3p+1}{2}}\binom{4p-1}{p-3}\frac{3p+1}{2}x^{\frac{3p-1}{2}}.$$

Upon addition:

$$B(x)\equiv\frac{d}{p}\frac{p+1}{2}x^{\frac{3p-1}{2}}\left(\frac{1}{3}\binom{2p-1}{\frac{3p+1}{2}}\binom{4p-1}{p-3}-\binom{\frac{3p+1}{2}}{\frac{p+1}{2}}\binom{p-1}{\frac{p+1}{2}}\binom{2p-1}{p-3}\right) \mod p$$

$$B(x)\not\equiv 0 \Longleftrightarrow \frac{1}{3}\binom{2p-1}{\frac{3p+1}{2}}\binom{4p-1}{p-3}-\binom{\frac{3p+1}{2}}{\frac{p+1}{2}}\binom{p-1}{\frac{p+1}{2}}\binom{2p-1}{p-3}\not\equiv 0$$

$$\begin{aligned}
\Longleftrightarrow \quad & \frac{(\frac{3p+1}{2})!(p-1)!(2p-1)!}{(\frac{p+1}{2})!^2p!(\frac{p-3}{2})!(p-3)!(p+2)!}-\frac{1}{3}\frac{(2p-1)!(4p-1)!}{(\frac{3p+1}{2})!(\frac{p-3}{2})!(p-3)!(3p+2)!}\\
= \quad & \frac{1}{(\frac{p-3}{2})!(p-3)!}\left(\frac{(\frac{3p+1}{2})!(p-1)!(2p-1)!}{(\frac{p+1}{2})!^2p!(p+2)!}-\frac{1}{3}\frac{(2p-1)!(4p-1)!}{(\frac{3p+1}{2})!(3p+2)!}\right)\\
= \quad & \frac{1}{(\frac{p-3}{2})!(p-3)!}\left(\frac{1}{2(\frac{p+1}{2})!}-\frac{1}{6(\frac{p+1}{2})!}\right)\not\equiv 0 \mod p.
\end{aligned}$$

□

4.3 Irrationality measure of CDF

Lemma 4.1 (Upper bound of irrationality measure)**.** *Suppose for two infinite sequences of positive integers, $[A_n],[B_n]$ and a positive, explicit real number t, the following inequality relationship*

$$|A_nx-B_n|\le\frac{C}{A_n^t}$$

Where C is a constant independent of n, holds for all $n>0$. Then x is irrational; its irrationality measure has an upper bound of $1+\frac{1}{t}$.

This folklore lemma has appeared in people's work frequently. However, we cannot specify the earlier version of this lemma or its inventor.

In this section, we will use this lemma to find an upper bound for the irrationality measure of CDF.

$$|A_{(m,n)}(x)f(x)-B_{(m,n)}(x)|=|R_{(m,n)}(x)|\sim o\left(\frac{1}{(m!)^{1-\epsilon}}\right);$$

$$A_{(m,n)}(x) = \sum_{i=0}^{n}(-1)^i \binom{n}{i}\binom{2m+2i+1}{2m}\frac{(m+i)!}{m!}x^{n-i},$$

$$i \leq n;$$

$$|A_{(m,n)}(x)| < \sum_{i=0}^{n}\binom{n}{i}\binom{m+i}{m}\binom{2m+2i+1}{2m}n!x^{n-i}.$$

When $n \to \infty$,

$$|A_{(m,n)}(x)| < (n!)^{1-\epsilon}.$$

$$n = \frac{3p+1}{2}, m = \frac{p-3}{2}, n = 3m+5.$$

$$\frac{1}{|A_{(m,n)}(x)|} > (\frac{1}{(3m)!})^{1+\epsilon}.$$

Hence,

$$|R_{(m,n)}(x)| < \frac{C}{|A_{(m,n)}(x)|^{\frac{1}{3}(1-\epsilon)}}$$

$$t = \frac{1}{3}.$$

The results of this section directly leads to the upper bound of the irrationality measure of CDF of normal distribution.

Theorem 4.2 (irrationality of CDF of normal distribution)**.** *For the cumulative distribution function of normal distribution,*

$$f(x) = \sum_{k=0}^{\infty}\frac{x^k}{(2k+1)k!},$$

For every rational $x \neq 0$, the value of $f(x)$ is irrational; its irrationality measure has an upper bound 4.

5 Irrationality of a Special Form of Confluent Hypergeometric Function

In this section, we will study a special formal power series:

$$f(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!(bk+s)(bk+s+1)...(bk+t)}$$

Where b, s, t are positive integers, $g.c.d(b,t) = 1$. We will use a similar method of explicitly constructing the Padé approximants, and we will generalise the "mod-p" proof for non-vanishing of remainder terms, to complete the proof of irrationality to this type of function.

Notice that when $b = 1$,

$$f(x) = \frac{1}{s(s+1)...t} {}_1F_1\left[\begin{matrix} s \\ t+1 \end{matrix}; x\right];$$

When $t = s$,

$$f(x) = \frac{1}{b} {}_1F_1\left[\begin{matrix} \frac{t}{b} \\ \frac{t}{b}+1 \end{matrix}; x\right].$$

(notice that in this case, the limiting condition $g.c.d(b, t) = 1$ is not necessary as the single term $(bk + t)$ in the denominator can be reduced.)

5.1 Padé approximants

We will first construct a general form of Padé approximants to $f(x)$:

$$A(x)f(x) - B(x) = R(x)$$

Similar to the previous sections, to eliminate the $k!$ in the denominator of $f(x)$, we will still take the term $\frac{(m+i)!}{m!}$ in the numerator of $A(x)$; to eliminate the terms $(bk + s)(bk + s + 1)...(bk + t)$, we will take $\binom{b(m+i)+t}{bi+s}$.

$$A(x) = d \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(m+i)!}{m!} \binom{b(m+i)+t}{bi+s} x^{n-i};$$

$$B(x) = \sum_{k=0}^{n-1} \sum_{i=n-k}^{n} (-1)^i \binom{n}{i} \binom{m+i}{i} \binom{b(m+i)+t}{bm} \frac{i!}{(k-n+i)!} \frac{d}{(b(k-n+i)+s)...(b(k-n+i)+t)} x^k$$

$$R(x) = \frac{x^{m+n}}{m!} \sum_{k=0}^{\infty} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{b(m+i)+t}{bm} \frac{(m+i)!}{(k+m+i)!} \frac{d}{(b(k+m+i)+s)...(b(k+m+i)+t)} x^k$$

Where $d = l.c.m[s, s+1, s+2, ..., b(n-1)+t]$. Following the same argument, our aim is to prove $B(x) \not\equiv 0(\mod p)$ for specific p. Let

$$m = \left\lfloor \frac{p}{b} \right\rfloor, n = p + \frac{p-t}{b} + 1,$$

Notice that $g.c.d(b,t)=1$, so that there are infinite number of prime number p satisfy the condition $\frac{p-t}{b}$ is an integer, by Dirichlet prime number theorem.

We have the following mod p lemma for $B(x)$ when p is chosen as above.

Lemma 5.1. $B(x) \not\equiv 0 \mod p$.

Proof. We consider the following subcases,

1. When $p \nmid b(k-n+i)+t$:

$$p \Big| \frac{d}{b(k-n+i)+t}.$$

2. When $p|b(k-n+i)+t$:

   (a) If $b(k-n+i)+t=p$:

$$k-n+i<p; i \geq \frac{p-t}{b}+(n-k) \geq \frac{p-t}{b}+1.$$

   If $i \geq p$:

$$p \Big| \frac{i!}{(k-n+i)!}.$$

   If $\frac{p-t}{b}+2 \leq i < p$:

$$p \Big| \binom{p+\frac{p-t}{b}+1}{i}.$$

   The only term left not divisible by p is $i=\frac{p-t}{b}+1$, $k=n-1$:

$$p \nmid (-1)^{\frac{p-t}{b}+1} \binom{p+\frac{p-t}{b}+1}{p} \binom{\lfloor \frac{p}{b} \rfloor + \frac{p-t}{b}+1}{\frac{p-t}{b}+1} \binom{p+b\lfloor \frac{p}{b} \rfloor}{p} (\frac{p-t}{b}+1) \frac{d}{p} x^{\frac{p-t}{b}+1}.$$

   (b) When $b(k-n+i)+t=lp$, $1<l<b$:

$$k-n+i=\frac{lp-t}{b}<p.$$

   If $\frac{p-t}{b}+2 \leq i < p$:

$$p \Big| \binom{p+\frac{p-t}{b}+1}{i}.$$

   If $i \geq p$:

$$p \Big| \frac{i!}{(k-n+i)!}.$$

(c) When $b(k-n+i)+t=(b+1)p$:

$$i=n=p+\frac{p-t}{b}+1; k=n-1=i-1.$$

Where i,k are both maximised.

$$p \nmid (-1)^{p+\frac{p-t}{b}+1}\binom{p+\lfloor \frac{p}{b}\rfloor+\frac{p-t}{b}+1}{p+\lfloor \frac{p}{b}\rfloor}\binom{(b+1)p+b\lfloor \frac{p}{b}\rfloor}{(b+1)p}(p+\frac{p-t}{b}+1)\frac{d}{(b+1)p}x^{p+\frac{p-t}{b}+1}.$$

3. When $p|(b(k-n+i)+s)...(b(k-n+i)+t-1)$: Notice that $b(k-n+i)+s<b(k-n+i)+s+1<...<b(k-n+i)+t-1<(b+1)p$. If $p-1\geq i\geq \frac{p-t}{b}+2$:

$$p|\binom{p+\frac{p-t}{b}+1}{i}.$$

If $p\leq i$:

$$k-n+i\leq p+1-s,$$
$$p|\frac{i!}{(k-n+i)!}.$$

We have shown that only two terms in $B(x)$ has a non-zero remainder mod-p. To show that upon addition, the two terms still produce a non-zero remainder, we will use the following lemma:

Lemma 5.2. *If $a\geq b$, $c\geq d$, $a\equiv b$, $c\equiv d \mod p$, p is a prime number;*

$$\binom{a}{b}\equiv\binom{c}{d} \mod p.$$

Proof. It is straight-forward that

$$\frac{(n+p)!}{(m+p)!}\equiv\frac{n!}{m!} \mod p.$$

Hence,

$$\frac{(n+p)!}{m!(n-m+p)!}\equiv\frac{n!}{m!(n-m)!} \mod p$$
$$\binom{n+p}{m}\equiv\binom{n}{m} \mod p.$$

Similarly,

$$\binom{n}{m+p}\equiv\binom{n}{m} \mod p.$$

Now we can complete the proof by induction. ∎

Notice that in (2.3) $i, k \mod p$ is taken the same value as (2.1). Using lemma 5.2, this will directly leads to:

$$\binom{p+\frac{p-t}{b}+1}{p}\binom{\lfloor\frac{p}{b}\rfloor+\frac{p-t}{b}+1}{\frac{p-t}{b}+1}\binom{p+b\lfloor\frac{p}{b}\rfloor}{p}(\frac{p-t}{b}+1)$$

$$\equiv \binom{p+\lfloor\frac{p}{b}\rfloor+\frac{p-t}{b}+1}{p+\lfloor\frac{p}{b}\rfloor}\binom{(b+1)p+b\lfloor\frac{p}{b}\rfloor}{(b+1)p}(p+\frac{p-t}{b}+1)$$

$$\equiv W \not\equiv 0 \mod p.$$

$$B(x) \equiv (-1)^{\frac{p-t}{b}+1}\frac{d}{p}+(-1)^{p+\frac{p-t}{b}+1}\frac{d}{(b+1)p})x^{p+\frac{p-t}{b}+1}W \equiv |\frac{d}{p}-\frac{d}{(b+1)p}|Wx^{p+\frac{p-t}{b}+1} \mod p$$

$$p \nmid B(x).$$

□

5.2 Estimation of irrationality measure

$$R(x)=\frac{x^{m+n}}{m!}\sum_{k=0}^{\infty}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{b(m+i)+t}{bm}\frac{(m+i)!}{(k+m+i)!}\frac{d}{(b(k+m+i)+s)...(b(k+m+i)+t)}x^k;$$

$$A(x)=d\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{(m+i)!}{m!}\binom{b(m+i)+t}{bi+s}x^{n-i}.$$

When $n, m \to \infty$:

$$|R(x)| < (\frac{1}{m!})^{(1-\epsilon)};$$

$$|A(x)| < (n!)^{(1+\epsilon)}.$$

Where

$$m=\left\lfloor\frac{p}{b}\right\rfloor, n=p+\frac{p-t}{b}.$$

For sufficiently large p, $bm < n < (b+2)m$.

$$|R_{(m,n)}(x)| < \frac{C}{|A_{(m,n)}(x)|^{\frac{1-\epsilon}{b+1}}},$$

$$t=\frac{1}{b+1};$$

Hence the upper bound of the irrationality of $f(x)$ is $b+2$.

Theorem 5.3 (Irrationality of a generalised form of confluent hypergeometric function). *For function defined as*

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!(bk+s)(bk+s+1)...(bk+t)},$$

where b, t, s are positive integers and $g.c.d(b, t) = 1$: when $x \neq 0$ is rational, the value of $f(x)$ is irrational, and its irrationality measure has an upper bound of $b+2$.

From this result, we can obtain the proof of irrationality and an upper bound of irrationality measures to some confluent hypergeometric functions. By shifting the value chosen for m, n to $m = p-1, n = 2p-1$, we can also apply this theorem a form of Ein (exponential integral) function, which can be written as

$$f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k!k}.$$

Which has an upper bound of irrationality measure of 3 ($b = 1, t = s = 0$).

6 Conclusion

In this paper, we have mainly used Padé approximation to study the irrationality of special function values.

Focusing on functions in a similar form of confluent hypergeometric functions, we have constructed the Padé-type approximants explicitly and derived an estimation to the growth rate of the remainder terms. We have generalised the proof of Maier to use a mod p approach to show that the remainder term is non-vanishing, which completed our proof for the irrationality of a generalised form of confluent hypergeometric function. Our explicit representation of approximants also provides a direct implication to the upper bound for the irrationality of the values of these functions at rational points. This result can be used to show the irrationality property of exponential function, exponential integral function, CDF of normal distribution, etc.

It is worth notice that our bound of irrationality measure considerably wide and has place for improvement, since our estimates to the growth rates of the denominators and remainders are not the most accurate. Although Euler's hypergeometric transformation can provide a good integration representations to the remainder terms, it is still difficult to estimate its growth rate due to the alternative signs of each terms produced by the factor $(-1)^i$. Meanwhile, our research result is limited to the confluent-type hypergeometric functions with specific parameters, which indicates possible direction of future study including improvement on upper bound of irrationality measures, as well as studying on more generalised forms of functions.

Another important ingredient used in our proof is the mod-p method to achieve non-vanishing of remainder terms. This involves special choices of degrees of polynomials related to prime numbers p and detailed study of mod-p results of some products of binomial coefficients. We hope to carry out this to p-adic results of these Padé type approximations.

References

[1] F. Beukers. A refined version of the Siegel-Shidlovskii theorem. *Annals of mathematics*, pages 369–379, 2006.

[2] G. V. Chudnovsky. Padé approximations to the generalized hypergeometric functions. I. *J. Math. Pures Appl.*, 58:445–476, 1979.

[3] C. Davis. Rational approximations to e. *Journal of the Australian Mathematical Society*, 25(4):497–502, 1978.

[4] W. Maier. Potenzreihen irrationalen grenzwertes. *Journal für die reine und angewandte Mathematik*, 1927(156):93–148, 1927.

[5] V. Salikhov. On the measure of irrationality of the number π. *Mathematical Notes*, 88, 2010.

[6] C. L. Siegel. *Transcendental Numbers.(AM-16)*. Princeton University Press, 2016.

[7] A. Van der Poorten. A proof that Euler missed... *The Mathematical Intelligencer*, 1(4):195–203, 1979.